Mathematical Reviews

Vol. 4, No. 1 January, 1943 pp. 1–32

TABLE OF CONTENTS

THE REPRINT SERVICE

The subscribers to MATHEMATICAL REVIEWS can obtain, either on microfilm or as photoprint, the complete text of any article of which an abstract is printed, except books, or material of which the reproduction is prohibited by copyright law. The microfilm requires some sort of reading machine or projector in which standard 35 mm. double perforated film can be viewed. Only *positive* prints are furnished; that is, the projected image appears as black words on a white background. Photoprints, on the other hand, are on sheets 10" by 14", and are similar to photostats. They can be read directly without the aid of any reading device. They also are furnished as positives, that is, black words on a white background.

Articles not abstracted in the MATHEMATICAL REVIEWS will be copied at the rates mentioned below provided the material is available in the library of Brown University. All microfilm copies of articles other than those with MF numbers will be supplied as negatives.

Cost

The cost of microfilm is 2¢ per exposure, *each exposure* (except in the case of a very few unusually large journals) *containing two pages of text*.

Photoprints cost 16¢ per sheet, *each sheet* (except in the case of unusually large journals) *containing two pages of text*.

The minimum charge for any one order (but not for any one item when several items are ordered simultaneously) is 50¢.

How to Order

The process of ordering either microfilm or photoprint may be made clear by considering an example. Each abstract bears a serial number easily recognized by the fact that it is always preceded by the abbreviation "MF". A subscriber who wishes the text of item 18 would first note that the original article occupied pages 590 to 608. It therefore covers 19 pages, and will require ten exposures of microfilm, or ten sheets of photoprint for its reproduction. Hence the microfilm charge for this particular item would be 20¢, and the photoprint charge $1.60. The charges so computed apply on any order totaling 50¢ or more; if the total is less than 50¢, the charge for the entire order will be 50¢.

In ordering:

1. Give your name and address.
2. State the serial number of the article or articles you wish.
3. Say whether you wish microfilm or photoprint.
4. Enclose cash for the amount of the order, either in currency, stamps, or by check.
5. Mail to MATHEMATICAL REVIEWS, Brown University, Providence, Rhode Island.

In the case of large journals, only one page of which can be accommodated on a microfilm exposure or a photoprint sheet, you will be notified of the extra charge before the order is filled, unless you have indicated on your order that the reprint is desired even at the higher price.

Why Cash Must Accompany Order

Individual orders for either microfilm or photoprint are so small that if credit books were kept and bills rendered, the extra clerical expenses would be a large proportion of the total. By requiring payment in advance, these charges have been eliminated, and the price to the customer correspondingly reduced. However, libraries and institutions which make extensive use of the service may be granted the privilege of payment at stated intervals.

ONE-SIDE EDITION OF MATHEMATICAL REVIEWS

In order to supply persons interested in making card files for subject indices of their own fields, or to add bibliographical remarks to the REVIEWS in the future, there will be available an edition of MATHEMATICAL REVIEWS printed on only one side of the paper. This special edition may be obtained for an additional payment of $1.00 (that is, $14.00 for subscriptions, and $7.50 for members of sponsoring organizations). This edition will be folded but not stitched. Subscriptions for this edition should be entered as soon as possible since only a limited number can be printed. For the first four issues, only the two-side edition is available. A current regular subscription can be changed to the one-side subscription by informing MATHEMATICAL REVIEWS and paying $1.00.

MATHEMATICAL REVIEWS

Published monthly, except August, by

THE AMERICAN MATHEMATICAL SOCIETY, Prince and Lemon Streets, Lancaster, Pennsylvania

Sponsored by

THE AMERICAN MATHEMATICAL SOCIETY
THE MATHEMATICAL ASSOCIATION OF AMERICA
ACADEMIA NACIONAL DE CIENCIAS EXACTAS, FISICAS Y NATURALES DE LIMA
HET WISKUNDIG GENOOTSCHAP TE AMSTERDAM
THE LONDON MATHEMATICAL SOCIETY
UNION MATEMATICA ARGENTINA

Editorial Office

MATHEMATICAL REVIEWS, Brown University, Providence, R. I.

Subscriptions: Price $13 per year ($6.50 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions should be addressed to MATHEMATICAL REVIEWS, Lancaster, Pennsylvania, or Brown University, Providence, Rhode Island.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 538, P. L. and R. authorized November 7, 1940.

Vo

Mi
F
[
N
of r
cas
eve
div
inv
and
gro
gro
sub
con

Mi
i
(
I
tw
tim
tio
reg
pr
wi

M

th
of
m
th
w
l i
of

P

a
by
o
pa
S
H
T
to
th
S
sl
fo

Mathematical Reviews

Vol. 4, No. 1 JANUARY, 1943 Pages 1-32

THEORY OF GROUPS

Miller, G. A. Maximal Sylow subgroups of a given group. Proc. Nat. Acad. Sci. U.S.A. **28**, 80–83 (1942). [MF 6327]

Maximal Sylow subgroups of a solvable group can not be of more than two different orders. What happens in various cases is described. Then it is shown that a group G, in which every maximal subgroup is a Sylow group, has an order divisible by just two primes, and that either it has two invariant Sylow groups, or one is invariant with prime index and is also Abelian of type 1^m. Finally a study is made of groups in which every Sylow group is maximal. If such a group should contain other maximal subgroups (not Sylow subgroups) it would be insolvable and its order would contain each prime factor to at least the third power.

J. S. Frame (Meadville, Pa.).

Miller, G. A. Certain direct products of the groups of self-isometries. Proc. Nat. Acad. Sci. U.S.A. **28**, 141–144 (1942). [MF 6485]

It is shown that a dihedral group is the direct product of two of its proper subgroups if and only if its order is four times an odd number. Then certain abstract defining relations are given for the three groups of self-isometries of the regular solids, each of which is known to be the direct product of an invariant subgroup of movements of the body with a group of order 2. *J. S. Frame.*

Miller, G. A. Some deductions from Frobenius's theorem. Proc. Nat. Acad. Sci. U.S.A. **28**, 251–254 (1942). [MF 6681]

G. Frobenius announced in 1895 that, if m is a factor of the order of a finite group G, then the number of operators of G, including the identity, whose orders divide m is a multiple of m. The author discusses some implications of this theorem and proves that the number of operators of G whose orders are powers of a prime p is $(p+l(p-1))p^m$, where l is an integer not less than -1, and p^m is the highest power of p which divides G. *J. S. Frame* (Meadville, Pa.).

Piccard, Sophie. Sur les bases du groupe symétrique. Mathematica, Timişoara **17**, 147–166 (1941). [MF 6737]

Two permutations S, T of the symbols $1, 2, 3, \cdots, n$ are said to be primitive in this paper if the group generated by them is transitive on the n symbols and primitive in the ordinary sense. The author has investigated in a former paper the conditions under which certain special forms of S, T generate the symmetric group [Comment. Math. Helv. **12**, 130–148 (1939); these Rev. **1**, 161]. Here, taking $T=(a, b, c, d)$, where a, b, c, d are any four integers from 1 to n, she extends one of her former theorems and proves that for $n\geqq 7$ the necessary and sufficient condition that S, T should generate the symmetric group is that they should be primitive. A very considerable number of possible forms for the substitution S have to be considered.

G. de B. Robinson (Ottawa, Ont.).

Brauer, Richard. On groups whose order contains a prime number to the first power. I. Amer. J. Math. **64**, 401–420 (1942). [MF 6441]

In this paper, which is dedicated to I. Schur, the theory of the modular representations of a finite group, as developed by the author during the past few years, is applied to the study of the table of ordinary characters of a group $\mathfrak{G}$ whose order is divisible by at least one prime p to the first power only. While it might be supposed that this restriction would materially limit the interest of the groups considered, the author points out that of Dickson's 78 simple groups of order less than 10^9 there is only one which does not fall into this class. In consequence of this restriction the Sylow-subgroup $\mathfrak{P}$ corresponding to the prime p is of simple structure as is also that of its normalizer $\mathfrak{N}$. Though the arguments required to discuss the relation between the ordinary characters and the characters mod p are involved, the final conclusions are simple and elegant. I quote the following paragraphs from the author's introduction, noting that an element of the group $\mathfrak{G}$ is defined to be p-regular if its order is prime to p, otherwise p-singular; a class of p-regular (singular) elements is said to be p-regular (singular).

"Let K be the number of conjugate elements in $\mathfrak{G}$. The characters of $\mathfrak{G}$ can be arranged in the form of a matrix Z of degree K. Each row corresponds to a fixed character, each column to a fixed class of conjugate elements. We arrange the columns so that the p-regular classes come before the p-singular classes. Thus if

$$(^{**})\quad Z=\left(\begin{array}{cc} Z_1 & Z_2 \\ \underbrace{Z_3}_{p\text{-regular classes}} & \underbrace{Z_4}_{p\text{-singular classes}} \end{array}\right)\begin{array}{l}\}\text{ degrees prime to } p,\\ \}\text{ degrees divisible by } p,\end{array}$$

then we have first of all that $Z_4=0$. Our main result states that the matrix Z_2 is determined by the structure of the normalizer $\mathfrak{N}$ of the Sylow-subgroup $\mathfrak{P}$, only some $\pm$ signs remaining undetermined. In particular, if we replace $\mathfrak{G}$ by $\mathfrak{N}$ the matrix Z_2 remains essentially the same, only some of its rows having to be changed. In the case of the group $\mathfrak{N}$ the second row in (**) is missing. This implies that the number of characters of $\mathfrak{G}$ whose degree is prime to p is equal to the number of classes of conjugate elements of $\mathfrak{N}$. The values of the characters of $\mathfrak{G}$ for p-singular elements can be written down (apart from the $\pm$ signs which remain undetermined) provided that the characters of a certain subgroup $\mathfrak{W}$ are known. This $\mathfrak{W}$ is a subgroup of $\mathfrak{N}$ whose order is prime to p, and hence the construction of its characters can certainly be considered as a more elementary problem than the corresponding problem for $\mathfrak{G}$. As a matter of fact, in most of the applications of the theory developed here the group $\mathfrak{W}$ is of very simple structure and its characters can be written down at once. Very often $\mathfrak{W}$ is a cyclic group.

From a knowledge of the matrices Z_2 and Z_4 in (**) some information concerning Z_1 and Z_3 can be obtained. First the orthogonality relations for group characters can be applied. Secondly, we may obtain the values (mod p) of the characters for those classes whose elements commute with the elements of order p. In particular, this gives a set of new conditions for the degrees of the characters which in many cases is sufficient for the computation of the degrees. Finally, from our knowledge of Z_2 and Z_4 we obtain conditions for the multiplication of the characters which also can be used to gain new information concerning Z_1 and Z_3."

G. de B. Robinson (Ottawa, Ont.)

Brauer, Richard. On groups whose order contains a prime number to the first power. II. Amer. J. Math. **64**, 421–440 (1942). [MF 6442]

It was proved by Blichfeldt that the order g of a primitive unimodular linear group $\mathfrak{G}$ on n variables is divisible by no prime p greater than $(2n+1)(n-1)$. From his enumerations of these groups for $n=4, 5$ it would appear that this upper limit for p is far from being the "best possible." The author now proves that, for primes p which divide g to the first power only, $p \leqq 2n+1$. Equality obtains only when $\mathfrak{G}$ considered as a collineation group is isomorphic with $LF(2, p)$. The proof of this important result is based upon the preceding paper [see the preceding review]. As a consequence of the properties of the characters there deduced, it is possible to limit the problem to proving the theorem for those groups $\mathfrak{G}$ which are identical with their commutator subgroups. In spite of this limitation the argument is complicated and cannot be described here.

G. de B. Robinson (Ottawa, Ont.).

Best, Ernest and Taussky, Olga. A class of groups. Proc. Roy. Irish Acad. Sect. A. **47**, 55–62 (1942). [MF 6449]

The authors term a t-group any group G which satisfies the following condition: The subgroup S of G is a normal subgroup of G if (and only if) there exists a normal subgroup T of G such that S is a normal subgroup of T. It is shown that a group of prime power order is a t-group if, and only if, it is Abelian or Hamiltonian. Furthermore, it is shown that a group is a t-group if it may be generated by elements $p_1, \cdots, p_n, q$ subject to the relations: $p_i^m = q^n = 1$, $p_i p_j = p_j p_i$, $p_i q = q p_i^r$, where $(n, m) = 1$, $r^n \equiv 1 \bmod m$. Thus all the groups whose Sylow subgroups are cyclic are t-groups.

R. Baer (Urbana, Ill.).

Levi, F. W. On the number of generators of a free product, and a lemma of Alexander Kurosch. J. Indian Math. Soc. (N.S.) 5, 149–155 (1941). [MF 6977]

This paper contains a proof of the following important theorem: if the group G is the free product of the groups A and B, if A and B are both different from 1, and if it is possible to generate G by two elements, then both A and B are cyclic. Various applications are given. *R. Baer.*

Suetuna, Zyoiti. Über die sich selbst assoziierten Charaktere der symmetrischen Gruppe. J. Reine Angew. Math. **183**, 92–97 (1941). [MF 6840]

Various types of self-associated characters of the symmetric group on n symbols are obtained which exhaust the possibilities for $n \leqq 25$. The discussion could have been clarified by utilizing the simple properties of the Young tableaux. *G. de B. Robinson* (Ottawa, Ont.).

Herring, Conyers. Character tables for two space groups. J. Franklin Inst. **233**, 525–543 (1942). [MF 6674]

The construction of character tables for the irreducible representations of the space groups of crystal lattices is comparatively easy when each lattice point has the full symmetry of the point group. In this paper character tables are given for the two most important space groups of the more difficult case, where no point of the space has the symmetry of the point group, namely, the types $\mathfrak{D}_{6h}^4$ (close-packed hexagonal) and $\mathfrak{D}_h^7$ (diamond type). The author uses the notation of F. Seitz. It is shown that the construction of the character of an irreducible representation is nontrivial only if the wave vector $\mathbf{k}$ of the representation terminates at a point of symmetry or on a line of symmetry of the Brillouin zone of the lattice. In this case the character is that of the corresponding representation of the factor group G^k/T^k, where G^k is the subgroup of the space group which takes each element of the representation space with wave vector $\mathbf{k}$ into an element with wave vector equivalent to $\mathbf{k}$, and T^k consists of all translations of the space group whose scalar product with $\mathbf{k}$ is a multiple of 2π. These factor groups are all finite, and their primary characters are computed by means of Burnside's theorem. The results are listed in nine tables for the close-packed hexagonal lattice, and four further tables for the diamond type lattice. Most of these tables contain several primary characters, each associated with one of the points of symmetry or lines of symmetry of the Brillouin zone. *O. Frink.*

Halmos, Paul R. and Samelson, H. On monothetic groups. Proc. Nat. Acad. Sci. U.S.A. **28**, 254–258 (1942). [MF 6682]

In a topological group G, let $D(G)$ be the set of all $x \epsilon G$ such that the subgroup $\{x^n\}$ $(n=0, \pm 1, \pm 2, \cdots)$ generated by x is everywhere dense in G. The authors call G "monothetic" if $D(G)$ is nonvoid; G must then be Abelian, and, if locally compact and not isomorphic to the additive group of integers, must be compact. It is shown that the compact Abelian group G is "monothetic" if and only if: (a) G is "separable" (there exists an everywhere dense enumerable subset of G); (b) the group of the characters of finite order of G is isomorphic to a subgroup of the additive group of rational numbers mod 1. Furthermore, if G is Abelian, compact, connected and satisfies the second countability axiom (that is, has an enumerable character group), $D(G)$ has the Haar measure 1, that is, almost every element of G belongs to $D(G)$. In a totally disconnected group, the measure of $D(G)$ may take any value between 0 and 1; if the second countability axiom is not satisfied, it is asserted that $D(G)$ need not be measurable. All these results are direct consequences of the duality theory of Abelian groups.

A. Weil (Bethlehem, Pa.).

Chevalley, Claude. An algebraic proof of a property of Lie groups. Amer. J. Math. **63**, 785–793 (1941). [MF 5626]

Let L be a Lie algebra over the complex field. An element h of L is regular if the number of zero roots of the characteristic polynomial of the matrix representing it in the adjoint representation is minimal. If this minimal number is l, it is known that h may be imbedded in a nilpotent subalgebra, called a Cartan algebra, of dimensionality l. The author proves the theorem: If N_0 and N_1 are any two Cartan subalgebras of L then there exists an operation of the adjoint group of L transforming N_0 into N_1. As the author remarks, this theorem is an analogue of Sylow's theorem on the

conjugacy of Sylow subgroups of a finite group. The special case where L is semisimple had been proved previously by Weyl using analytic methods and by Weil by topological methods. The present proof uses the methods of algebraic geometry. Of particular importance in the proof is the use of Plückerian coordinates and the use of a type of specialization lemma. *N. Jacobson* (Chapel Hill, N. C.).

Montgomery, Deane and Zippin, Leo. A theorem on Lie groups. Bull. Amer. Math. Soc. **48**, 448–452 (1942). [MF 6712]

If G^* is a compact subgroup of a Lie group G, Cartan [La Théorie des Groupes Finis et Continus et l'Analysis Situs, Mémor. Sci. Math., vol. 42, Gauthier-Villars, Paris, 1930, p. 43] has shown that the space M of cosets G/G^* has a Riemann metric for which G is a group of isometries. Using a property of geodesics in convex spheres of M and the isometric character of G, it is here proved that there is in G an open set O containing G^* such that H a subgroup of G and $H\subset O$ implies the existence of $g\epsilon G$ such that $g^{-1}Hg\subset G^*$. (For G^* invariant this theorem is an immediate consequence of the fact that Lie groups cannot have arbitrarily small subgroups.) Using results due to van Kampen [Ann. of Math. (2) **37**, 78–91 (1936), in particular p. 88; Amer. J. Math. **58**, 177–180 (1936)], it is further shown that, if G is a compact connected finite-dimensional group of which G^* is a closed connected subgroup, there is an open set $O\supset G^*$ such that H a closed connected group in O implies H is a transform of a subgroup of G^*.

W. W. Flexner (Ithaca, N. Y.).

Hopf, H. und Samelson, H. Ein Satz über die Wirkungsräume geschlossener Liescher Gruppen. Comment. Math. Helv. **13**, 240–251 (1941).

The authors prove that a manifold W which admits a compact transitive Lie group G of transformations has a nonnegative Euler characteristic. This theorem is established as a consequence of the following two propositions: (a) If a transformation f of G has only a finite number n of fixed points then n is equal to the Euler characteristic $\chi(W)$ of W. (b) There exists a transformation f of G which has only a finite number of fixed points.

The proof of (a) is quite simple (it is enough to show that every fixed point of f has the index $(-1)^d$, where d is the dimension of W). The proof of (b) is more difficult and is based on the consideration of Abelian subgroups of G. Let T be a maximum toroidal subgroup of G (that is, a toroidal subgroup which is not contained in a toroidal subgroup of higher dimension) and let a be a generating element of T (that is, an element whose powers form a set dense in T); the authors show that the transformation corresponding to a satisfies the condition of (b). The difference between the cases $\chi(W)>0$ and $\chi(W)=0$ is discussed. It turns out that the latter case is characterized by the occurrence of one-parameter-subgroups in G which yield streamlines free of singularities. *W. Hurewicz.*

Smith, P. A. Everywhere dense subgroups of Lie groups. Bull. Amer. Math. Soc. **48**, 309–312 (1942). [MF 6411]

The author proves the following theorem: Let G be a simple Lie group of dimension r greater than one and let G_1 be a proper subgroup filling G densely. There exists at least one proper closed Lie subgroup H of G such that those left- (right-) cosets of H which fail to meet G_1 fill G densely. For H one may take any closed proper Lie subgroup of G whose central is nondiscrete and contains an arbitrarily chosen point p in $G_1\cap U$, U being any given canonical nucleus of G. It is proved, incidentally, that if an everywhere dense subgroup of a simple Lie group contains an analytic arc then the subgroup is in reality the whole group.

D. Montgomery (Princeton, N. J.).

Smith, P. A. Stationary points of transformation groups. Proc. Nat. Acad. Sci. U.S.A. **28**, 293–297 (1942). [MF 6954]

Let (G, S) be a continuous group of topological transformations of a space S into itself homomorphic to a topological group G, and let $\sigma(G, S)$ be the set of points of S which are fixed under every element of (G, S). Let (n, p), $n>0$, be a compact finite-dimensional space having the same homology groups mod p as the n-sphere; let $(0, p)$ be a pair of points, $(-1, p)$ be the empty set; let E_n be a Euclidean n-space, R_n the group of rotations in E_n. That $\sigma(G, (n, p))$ is always an (m, p), $-1\leqq m\leqq n$, is written $[G, n, p]$; that every (G, E_n) has a stationary point is written $[G, n]$. If S is locally Euclidean and the functions defining (G, S) are analytic relative to analytically connected coordinate systems in G and S, (G, S) is called analytic. Then (A) if p is prime and G is an Abelian group with order a power of p, $[G, n, p]$ for every n. (B) If G is a connected compact Abelian Lie group then, for every n, $[G, n]$ and, for p a prime, $[G, n, p]$. (C) If $(3, 2)$ is locally Euclidean and G is a compact connected Lie group, then $[G, 3, 2]$ and $[G, 3]$. (D) In the analytic case $[R_n, n, 2]$ for $n\geqq 0$. (E) In the analytic case $[G, n, 2]$ for compact connected Lie group G and $n\leqq 5$. *W. W. Flexner.*

de Kerékjártó, B. Sur les groupes compacts de transformations topologiques des surfaces. Acta Math. **74**, 129–173 (1941). [MF 6514]

The main result of part I is the theorem that an infinite compact group of orientation-preserving transformations of a sphere into itself is topologically equivalent to the group R of rotations of the sphere or a one-parameter subgroup of R. It should be pointed out that this is a corollary of a theorem of Montgomery and Zippin concerning compact transformation groups in three-space [Amer. J. Math. **61**, 375–387 (1939)]. Part II contains an analysis of compact groups of transformations of a torus. It is shown that they are equivalent to certain groups definable by linear equations in bicircular coordinates on a torus, groups which can be obtained by starting with the bicircular translation group T or a one-parameter subgroup of T, and adjoining at most three of certain specified periodic linear transformations. *P. A. Smith* (New York, N. Y.).

Birkhoff, Garrett. Lattice-ordered groups. Ann. of Math. (2) **43**, 298–331 (1942). [MF 6464]

The author makes a systematic study of "lattice-ordered groups" or "l-groups." An l-group is a system which is at the same time a group (not necessarily commutative) and a lattice, the connection between the two types of structure consisting in the "homogeneity" of $\geqq$ with respect to $+$: $x\geqq y$ implies $a+x+b\geqq a+y+b$ for all a and b, where $\geqq$ is the lattice inclusion and $+$ the group composition. A large part of the discussion is devoted to noncommutative l-groups, which have not previously been studied. After a section on formalities, it is shown that an l-group must be

a distributive lattice, and every element except 0 is of infinite order. Denoting $a \cup 0$ by a^+, $a \cap 0$ by a^-, $a \cup -a$ by $|a|$, the author proves various relations of the form $|a|>0$ if $a\neq 0$, $|na|=|n||a|$ for any integer n; $|a|=a^+-a^-$. However the triangle inequality $|a+b|\leqq|a|+|b|$, although true for commutative l-groups, is not believed to hold in general. An l-ideal is defined as a normal subgroup which contains with each a all x such that $|x|\leqq|a|$, and l-ideals are shown to play the traditional role of ideals in determining homomorphisms.

In order that a commutative group be the additive group of an l-group, it need merely satisfy the condition that all its elements (except 0) have infinite order. Calling an l-group Archimedean if $a>0$ implies that a, $2a$, $3a$, $\cdots$ has no upper bound, one can show that any simply-ordered Archimedean l-group is isomorphic to a subgroup of the additive group of the real numbers, and hence is commutative; on the other hand an Archimedean l-group may have a non-Archimedean homomorph. A commutative l-group which is simple must be a subgroup of the additive group of the real numbers. The nature of commutative l-groups is illuminated by the result that if a commutative l-group has a structure lattice of finite length then it can be built up from simple l-groups by successive cardinal (=direct) products and mixed ordinal (=lexicographic) products. Any commutative l-group is isomorphic to an l-subgroup of a cardinal product of simply ordered l-groups. An l-group is said to satisfy the chain condition if every nonvoid set of positive elements includes a minimal member; such l-groups turn out to have several interesting properties, including that of being commutative. An l-group is called complete (σ-complete) if every nonvoid (countable) bounded set has a supremum; any σ-complete l-group is Archimedean; infinite distributive laws can be demonstrated in any complete l-group. The paper concludes with a very interesting collection of fifteen unsolved problems (the sixteenth seems to have become lost). *H. Wallman* (Cambridge, Mass.).

Clifford, A. H. Matrix representations of completely simple semigroups. Amer. J. Math. **64**, 327–342 (1942). [MF 6435]

A completely simple semigroup is a simple semigroup containing a primitive, nonzero, idempotent element [see D. Rees, Proc. Cambridge Philos. Soc. **37**, 434–435 (1941); these Rev. **3**, 199]. This is the Kerngruppe of Suschkewitch [Comm. Soc. Math. Kharkov et Inst. Sci. Math. Ukraine (6) **6**, 27–38 (1933)]. Rees has shown that these completely simple semigroups can be represented by matrices over groups with a "zero" adjoined [Proc. Cambridge Philos. Soc. **36**, 387–400 (1940), Theorem 2.93; these Rev. **2**, 127].

In the present paper Clifford gives a construction for all matrix representations of matrix semigroups. This construction is in terms of the representations of the group G, of factorizations of a matrix H arising from the representation of G, and some arbitrary auxiliary matrices. Even a finite semigroup will, in general, have an infinite number of inequivalent representations of the same degree. The questions of proper extensions, normal forms, reduction and decomposition are taken up. In the final section the same methods are applied to the groupoids of Brandt [Math. Ann. **96**, 360–366 (1927); also Vierteljschr. Naturforsch. Ges. Zürich **85** Bleiblatt (Festschrift Rudolf Fueter), 95–104 (1940); these Rev. **2**, 218]. If one adjoins an element 0 to the groupoid G, defining $ab=0$ if ab was previously undefined, and $0g=0=g0$, then G becomes a completely simple semigroup. The representation of G by the methods of this paper turns out to be unusually straightforward. *H. Campaigne* (Washington, D. C.).

ANALYSIS

Theory of Sets, Theory of Functions of Real Variables

Hornich, Hans. Ergänzung und Berichtigung zur meiner Arbeit: "Über eine Zusammensetzung von Mengen." Jber. Deutsch. Math. Verein. **51**, 80–81 (1941).

The paper appeared in the same Jber. **50**, 105–111 (1940); cf. these Rev. **2**, 131.

Deknatel, J. Sur le lieu des points équidistants de deux ensembles. Bull. Soc. Math. France **68**, 41–52 (1940). [MF 6807]

The problem concerns the locus $F(E_1, E_2)$ of points equidistant from two given bounded planar sets E_1 and E_2, $E_1E_2=0$, which, without change in generality, may be taken as closed. The major result characterizes this locus as follows: $F(E_1, E_2)$ consists of a finite number of closed Jordan curves, or curves going into such curves by inversion. These curves possess two semitangents at each of their points. If P is a fixed point on one such curve, Q the point of this curve at distance s along the curve from P in a fixed direction, then $\alpha(s)$, the angle between the semitangents at Q, is a function of limited variation; conversely, if a finite set of curves having these properties is given, there exists a pair of closed bounded sets E_1, E_2, $E_1E_2=0$, such that $F(E_1, E_2)$ consists of the given curves. *H. Blumberg.*

Otchan, G. Sur une question liée au problème de Souslin. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 423–426 (1941). (Russian. French summary) [MF 6831]

The author summarizes the contents of his paper as follows:

"Il est connu que le problème de Souslin serait résolu par l'affirmative, si l'on avait résolu par l'affirmative le problème suivant (plus fort que celui de Souslin): Démontrer que l'hypothèse suivante a réellement lieu: Hypothèse totale de Souslin: Chaque espace de Hausdorff parfaitement normal et bicompact contient un espace dénombrable dense dans cet espace. À côté de cette hypothèse on peut énoncer une autre (au premier abord plus faible): Hypothèse locale de Souslin: Chaque espace de Hausdorff bicompact et parfaitement normal contient au moins un ensemble dénombrable qui n'est pas partout non dense. Le but de la présente note est de démontrer l'équivalence de ces deux hypothèses." *J. V. Wehausen* (Columbia, Mo.).

Jones, F. B. Measure and other properties of a Hamel basis. Bull. Amer. Math. Soc. **48**, 472–481 (1942). [MF 6716]

A Hamel basis is a set H of real numbers such that every real number ($\neq 0$) is uniquely expressible as a (finite) linear combination, with rational coefficients ($\neq 0$), of the elements of H. If M is a set of real numbers, $T(M)$ is defined

to be the set of all numbers $x+y-z$, x, y and z belonging to M; $T^2(M)=T(T(M))$, $T^3(M)=T(T^2(M))$, $\cdots$; $T^0(M)=M$. Then $D(M)$ is defined to be the set of all numbers $x-y$, $x\geqq y$, x and y belonging to M. A variety of properties are proved for a Hamel basis, including the following: If H is a Hamel basis, the interior Lebesgue measure of $T^n(H)$, and also of $D^n(H)$, is zero for every n, but the exterior measure of $T^n(H)$ is positive for some n, and likewise for $D^n(H)$. The Cantor ternary set contains a Hamel basis. For an analytic set A not to contain a Hamel basis, it is necessary and sufficient that $mT^n(A)$ and $mD^n(A)$ are zero for every n. No discontinuous solutions of $f(x)+f(y)=f(x+y)$ is continuous in an analytical set containing a Hamel basis. There exists a Hamel basis containing a perfect set.

H. Blumberg (Columbus, Ohio).

Sen, H. K. Darboux's property and its applications. Proc. Benares Math. Soc. (N.S.) 2, 17–23 (1940). [MF 6644]

A real function $f(x)$, defined in the interval I, is said to have the Darboux property, denoted by D, if, in every subinterval (α, β) of I, f takes on all values between $f(\alpha)$ and $f(\beta)$. Obviously, if $f(x)$ has an external saltus on any side of any point, it cannot have property D. The author shows, conversely, for a function f of Baire's first class that, if there is no point at which f has an external saltus on any side, then f has property D. A number of corollaries and related facts are deduced, and several near-lying applications are made.

H. Blumberg (Columbus, Ohio).

Rossier, Paul. Sur la construction de courbes tangentielles sans point. C. R. Soc. Phys. Genève 57, 231–234 (1940). [MF 6490]

Dual to the problem of constructing a curve without a tangent at any point is the problem of constructing a curve as a continuous family of straight lines such that the intersection of a pair of neighboring lines has no limit as the two lines approach a common limit. On the surface of a sphere the author starts with a segment of a great circle, constructs an equilateral triangle on its middle third and discards the base. This construction is repeated on each of the resulting four segments, and so on. The family of circles polar to the points of the limit of this construction solves the problem on the sphere. The problem in the plane is solved by carrying out the middle third construction in terms of a Cayley metric and the principle of duality.

L. W. Cohen.

Shukla, Parmeshwar Din. On the derivates of a function of Denjoy. Proc. Benares Math. Soc. (N.S.) 2, 1–16 (1940). [MF 6643]

This paper continues and finishes the detailed discussion of the derivates of A. Denjoy's nondifferentiable function, begun by the author in a previous paper [Proc. Benares Math. Soc. (N.S.) 1, 97–102 (1939); these Rev. 1, 207].

A. Rosenthal (Albuquerque, N. M.).

Maximoff, Isaie. Sur les fonctions dérivées. Bull. Sci. Math. (2) 64, 116–121 (1940). [MF 6785]

Let E be a (linear) set, x_0 a point, and ρ and Δ positive numbers. If $(\delta'\geqq 0)(\delta''\geqq 0)(0<\delta=\delta'+\delta''\leqq\Delta)$ implies

$$m(i_{x_0}E)\geqq\delta-\delta^2/\rho^2,$$

where i_{x_0} is the interval $(x_0-\delta', x_0+\delta'')$ and m designates Lebesgue measure, then x_0 is said to be a point of density (ρ, Δ) of E. Using this restricted notion of density for sets the author defines a restricted type of approximate continuity and obtains that each function approximately continuous in this restricted sense is a derived function. The lemma requires the condition $\sum_{p=2}^{\infty}\rho_{n_{p-1}}^{-2}<\infty$ and the conclusion should be

$$f(x_0)-\eta\leqq\lim_{\delta_1\to 0}\underline{F}(\delta_1)\leqq\lim_{\delta_1\to 0}\overline{F}(\delta_1)\leqq f(x_0)+\eta,$$

where $\underline{F}(\delta_1)$ is the inf and $\overline{F}(\delta_1)$ is the sup of numbers of the form

$$\delta_1^{-1}\int_{x_0-\delta_1'}^{x_0+\delta_1''} f(x)dx, \qquad \delta_1=\delta_1'+\delta_1''.$$

J. F. Randolph (Ithaca, N. Y.)

Morrey, Charles B., Jr. A correction to a previous paper. Duke Math. J. 9, 120–124 (1942). [MF 6346]

As the title indicates the present paper is concerned with a correction of a result previously stated [Theorem 8.8, Duke Math. J. 6, 187–215 (1940); these Rev. 1, 209]. An example is given to show that the result as stated is false and a new theorem is given, which, together with other results, justifies the applications made by the author to the calculus of variations. It is shown further that the theorem as originally stated is true for a modified definition of weak convergence.

M. R. Hestenes (Chicago, Ill).

Radó, Tibor. On semi-continuity. Amer. Math. Monthly 49, 446–450 (1942). [MF 7147]

This paper is an elementary exposition of the concept of semicontinuity. It begins with a brief study of semicontinuous functions of one variable and indicates the generalization of the notion of semicontinuity to functionals having domains in limit spaces and ranges in spaces possessing an order relation. It concludes by pointing the applications of the concept to the theory of Lebesgue integrations, the calculus of variations and point set theory.

C. B. Morrey, Jr. (Aberdeen, Md.).

Rosenthal, Arthur. On differentiation of integrals and approximate continuity. Bull. Amer. Math. Soc. 48, 414–420 (1942). [MF 6706]

This paper considers the problem of generalizing the theorem that the derivative of an integral is the integrand to the case of functions defined over a metric space, where the integral is in the Radon-Fréchet sense. As the author points out, a careful investigation may be expected to be necessary, since even in the case of a Euclidean base space it is essential to distinguish between the cases of bounded and unbounded integrands, and the choice of the sets, or intervals, with respect to which the differentiation is performed essentially alters the results.

Among the results obtained the following may serve as a sample. Let R be a metric space, Ψ a nonnegative totally additive finite set function defined for sets in a σ-field M of subsets of R. Having set up appropriate definitions of an "indefinitely fine system of sets" O belonging to M, and of approximate continuity and differentiation relative to O, the author shows that the integral of a Ψ-measurable and Ψ-bounded function $f(x)$ has f as its derivative at all points where f is approximately continuous.

J. A. Clarkson.

Froda, Alex. Mesures extérieure et intérieure des ensembles-image des fonctions multiformes ou uniformes de variables réelles. Bull. Soc. Math. France 68, 83–108 (1940). [MF 6809]

A function $f(P)$ of n real variables is defined on the n-dimensional Euclidean space, and Δ_n is an interval in this

space. The symbols m_e, m_i denote, respectively, the outer and inner Lebesgue measure, and m the measure of measurable sets; V is the set of values of $f(P)$ at a point P. If I is the image of Δ_n by means of the function $f(P)$, and if, at each point P of Δ_n, $m_eV\geqq k$ ($m_iV\leqq k$), $k>0$, then the $(n+1)$-dimensional measures of I satisfy $m_eI\geqq km\Delta_n$ ($m_iI\leqq km\Delta_n$). If E is any set, measurable or nonmeasurable, and I the image of E, and if, at each point P of E, $m_eV\geqq k$ ($m_iV\leqq k$) then $m_eI\geqq km_eE$ ($m_iI\leqq km_iE$).

An example is given showing that the converse of the first of these relations is not true. A function is constructed which is such that $m_eI\geqq km\Delta_n$, but $m_eV<k$ at each point of Δ_n. At a point P of the measurable set E let $V_e(P)$, $V_i(P)$ be, respectively, the values of m_eV, m_iV. If $V_e(P)$ and $V_i(P)$ are measurable functions then

$$m_eI\geqq \int_E V_e(P)dP\geqq \int_E V_i(P)dP\geqq m_iI.$$

If the $(n+1)$-dimensional set I is measurable, and if, at each point P of the measurable set E, $m_eV=m_iV=mV=V(P)$, then $mI=\int_E V(P)dP$. *R. L. Jeffery.*

Theory of Functions of Complex Variables

Wavre, Rolin. Sur l'intégrale de Cauchy étendue à une ligne ouverte. C. R. Soc. Phys. Genève 57, 78–79 (1940). [MF 6487]

If C is an open curve in the complex plane and $f(z)$ a holomorphic function on C, the author considers what conditions on $f(z)$ are implied if $\int_C f(z)dz/(z-x)\equiv 0$ in a domain D of the complex plane. This is possible only if C cuts itself, and if $f(z)$ is multiple valued. In the special case where C consists of a curve with one double point, the author obtains information about the branch points of f and its singularities at the end points of C; the question of existence of these functions $f(z)$ is not answered.
J. W. Green (Rochester, N. Y.).

Shah, S. M. Note on a theorem of Polya. J. Indian Math. Soc. (N.S.) 5, 189–191 (1941). [MF 6981]

The author shows, generalizing results of Pólya and Valiron, that for any entire function

$$\varliminf_{r\to\infty}\frac{n(r)}{\log M(r)}\leqq\varliminf_{r\to\infty}\frac{\log^+ n(r)}{\log r}.$$

In the statement of his lemma, $\overline{\lim}$ and $\underline{\lim}$ have (rather confusingly) been interchanged. *E. S. Pondiczery.*

Shah, S. M. On integral functions of integral or zero order. Bull. Amer. Math. Soc. 48, 329–334 (1942). [MF 6414]

Let $f(z)$ be a canonical product of integral order ρ and genus $\rho-1$. Then, with the usual notation,

$$\liminf_{r\to\infty}\frac{\log M(r,f)}{n(r,f)\varphi(r)}=0$$

if $\varphi(r)$ is positive and increasing; $\int^\infty dx/(x\varphi(x))$ converges; and, for some $\alpha(0<\alpha<1)$, $x^{-\alpha}\varphi(x)$ is monotonic for large x. The same result (without the additional restriction involving $x^{-\alpha}\varphi(x)$) was proved earlier by the author for functions of genus equal to the order [J. London Math. Soc. 15, 23–31 (1940); these Rev. 1, 307]. The result is extended to entire functions more general than canonical products, and some additional results of similar character are given.
R. P. Boas, Jr. (Chapel Hill, N. C.).

Shah, S. M. A theorem on integral functions of integral order. II. J. Indian Math. Soc. (N.S.) 5, 179–188 (1941). [MF 6980]

If $f(z)$ is a canonical product of order ρ and genus $\rho-1$, then (with the usual notation)

$$\varliminf_{r\to\infty}\frac{\log M(r)}{n(r)\varphi(r)}=0$$

if $\varphi(r)$ is positive, continuous, nondecreasing, and such that $\int^\infty[r\varphi(r)]^{-1}dr$ converges. The author has previously [J. London Math. Soc. 15, 23–31 (1940); these Rev. 1, 307, 400] proved the corresponding result for canonical products of genus and order ρ. *E. S. Pondiczery* (Princeton, N. J.).

Shah, S. M. Note on a theorem of Valiron and Collingwood. Proc. Nat. Acad. Sci. India 12, 9–12 (1942). [MF 7038]

The theorem of Valiron and Collingwood cited [J. London Math. Soc. 4, 210–213 (1929)] states that, for an entire function of positive order $\rho<1$,

$$\varliminf_{r\to\infty}\frac{\mu N(r,a)}{\log M(r^\mu)}\geqq 1 \tag{1}$$

for all $\mu<1-\rho$ and all a. The author constructs an example to show that this result is best possible in certain directions. If $0<\rho<1$, his example satisfies

$$\varliminf_{r\to\infty}\frac{RN(r,0)}{\log M(r^{1-\rho})}=0, \tag{2}$$

$$\varliminf_{r\to\infty}\frac{\psi(r)N(r,0)}{\log M(r/R)}=0, \tag{3}$$

and (3) with N replaced by n, where $\psi(x)>0$,

$$\limsup_{x\to\infty}[\log\psi(x)]/\log x\leqq\rho,\quad R=\exp[(\log r)/\theta(r)],$$

and $\theta(r)\uparrow\infty$ as $r\uparrow\infty$. The point of the example is that (2) shows that (1) fails if $\mu=1-\rho$ instead of $\mu<1-\rho$, even with a factor increasing almost as rapidly as r in place of the constant μ in the numerator, while (3) shows that (1) fails if the power r^μ is replaced by a (slightly more slowly increasing) function $\omega(r)$ such that $\log\omega(r)=o(\log r)$, even with a factor increasing as rapidly as $\log r$ in the numerator. An analogous result is given for a class of functions of zero order.
E. S. Pondiczery (Princeton, N. J.).

Selberg, Atle. Über einen Satz von A. Gelfond. Arch. Math. Naturvid. 44, 159–170 (1941). [MF 5788]

Gelfond has shown that, if $g(z)$ is an entire function such that $g(n)$, $g'(n)$, $\cdots$, $g^{p-1}(n)$ are all integers when n is any nonnegative integer, then

$$|g(z)|<Ae^{\Theta|z|},\quad \Theta<p\log(1+e^{(1-p)/p}),$$

implies that $g(z)$ is a polynomial. In case $p=1$, the example $g(z)=2^z$ shows that this theorem is "best possible." The author shows that, for $p>1$, the bound on Θ, $p\log(1+e^{(1-p)/p})$, can be increased. In particular, as $p\to\infty$ the Gelfond bound behaves like $p\log(1+1/e)$, whereas the author's behaves like $p\log\omega$, where $\omega>1+1/e$. The author uses a more general interpolation polynomial than did Gelfond in carrying out the proof. *N. Levinson.*

Selberg, Atle. Über ganzwertige ganze transzendente Funktionen. II. Arch. Math. Naturvid. 44, 171–181 (1941). [MF 5789]

[Part I appeared in Arch. Math. Naturvid. 44, 45–52 (1941); cf. these Rev. 2, 356.] The author proves the fol-

lowing theorem: Let $f(z)$ be an entire function and let $M(r)$ be the max $|f(z)|$ on $|z|=r$. Let $f(n)$ be an integer for every integer value of n, $-\infty<n<\infty$. Then if

$$(*)\qquad \overline{\lim}\,\frac{\log M(r)}{r}\leq\log\frac{3+\sqrt{5}}{2}+2\cdot 10^{-6},$$

$f(z)$ must be of the form

$$P_1(z)\left(\frac{3+\sqrt{5}}{2}\right)^z+P_2(z)\left(\frac{3+\sqrt{5}}{2}\right)^{-z}+P_3(z),$$

where P_1, P_2 and P_3 are polynomials. This generalizes the result of Carlson which in place of (*) requires that

$$\overline{\lim}\,\frac{\log M(r)}{r}\leq\log\frac{3+\sqrt{5}}{2}.$$

The author uses difference and interpolation methods.

N. Levinson (Cambridge, Mass.).

af Hällström, Gunnar. Zwei Beispiele ganzer Funktionen mit algebraischem Höchstindex einer Stellensorte. Math. Z. 47, 161–174 (1941). [MF 6816]

The entire functions

$$c(z)=\prod_{k=1}^{\infty}\cos(z/k),\quad s(z)=\prod_{k=1}^{\infty}(\sin(z/k))/(z/k),$$

are studied in the light of the Nevanlinna theory of meromorphic functions. From this point of view the results of principal interest are these: both $c(z)$ and $s(z)$ are examples of a meromorphic function with index of ramification $\mu(0)$ equal to one; $c(z)$ and $s(z)$ are of maximal type of order one. The two functions are studied in detail. The zeros of the derivatives are located; the asymptotic values are determined; and the structures of the associated Riemann surfaces are given.

M. H. Heins (Chicago, Ill.).

Montel, Paul. Sur les valeurs algébriques d'une fonction entière ou méromorphe. J. Math. Pures Appl. (9) **20**, 305–324 (1941). [MF 6836]

The paper contains detailed proofs, together with special examples, of results announced earlier [C. R. Acad. Sci. Paris **211**, 217–220, 370–374 (1940); these Rev. **3**, 82].

M. S. Robertson (New Brunswick, N. J.).

Biggeri, Carlos. On exceptional values of analytic functions. Bol. Mat. 15, 9–13 (1942). (Spanish) [MF 6602]

The author announces, without proof, a number of theorems about Julia lines of entire functions which, he states, follow from his earlier results [Bol. Mat. **14**, 264–265 (1941); these Rev. **3**, 201]. For example, a line through the origin is a Julia line for $f(z)$ if $|f(z)|+|f'(z)|^{-1}$ is bounded for all points on the line of sufficiently large absolute value, or if on the line

$$\liminf_{|z|\to\infty}\frac{(|f(z)|+a)^7}{|zf'(z)|^6}=0$$

for every positive a. More general results of similar character deal with various sets of points on which $f(z)$ takes every value with at most one exception.

R. P. Boas, Jr.

Biggeri, Carlos. On the uniformization of analytic functions. Bol. Mat. 15, 49–56 (1942). (Spanish) [MF 6942]

The author discusses the uniformization of some functions which arise, in certain proofs of Picard's theorem on essential singular points, from assuming the theorem to be false. It seems to the reviewer that the statement at the foot of p. 50, from which the author starts, needs further justification except for entire functions.

R. P. Boas, Jr.

Rosenblatt, Alfred. On the coefficients of univalent series. Actas Acad. Ci. Lima 4, 145–155 (1941). (Spanish) [MF 6611]

G. M. Golusin has shown [Rec. Math. [Mat. Sbornik] N.S. **3**(45), 321–330 (1938)] that for the univalent function $f(z)=z(1+a_1z+\cdots)$, $|z|<1$, if $a_1=\cdots=a_{n-1}=0$, $n\geqq 2$, then $|a_k|\leqq 2/k$, $k=n,\cdots,2n-1$. By means of easy transformations the author points out as a consequence that for the odd univalent function $g(z)=z(1+b_2z^2+\cdots)$, $|z|<1$, if $b_2=\cdots=b_{2(n-1)}=0$, then $|b_{2k}|\leqq 1/k$, $k=n,\cdots,2n-1$. Extremizing functions are given. Further results are obtained for the case $n=2$.

E. F. Beckenbach.

Macintyre, A. J. and Wilson, R. Some converses of Fabry's theorem. J. London Math. Soc. 16, 220–229 (1941). [MF 6617]

Let $f(z)=c_0+c_1z+c_2z^2+\cdots+c_nz^n+\cdots$ have a radius of convergence unity. Fabry's theorem states that if $\lim_{n\to\infty}c_n/c_{n+1}=1$ then $z=1$ is singular for $f(z)$. The main interest of the author's paper is to prove some converses of Fabry's theorem. Following Pólya the author describes the singularity $z=1$ as easily approachable if $f(z)$ is regular in the neighborhood of $z=1$ throughout an angle $-\alpha<\arg(1-z)<\alpha$, $\alpha>\pi/2$. Likewise he calls the singularity almost isolated if $\alpha=\pi$, so that the region of regularity extends to the whole neighborhood of $z=1$ except for the real axis $z>1$. The author supplements these definitions of Pólya by saying that the singularity is virtually isolated if α can be taken to be greater than π. The author's main result may then be stated that, if $f(z)$ has on its circle of convergence a unique singularity at $z=1$, and ϵ is any positive number, then the inequality $|(c_{n+1}/c_n)-1|<\epsilon$ is satisfied (a) for a sequence of n of maximal density unity; (b) for a sequence of n of upper density unity if the singularity is easily approachable; (c) for a sequence of n of density unity if the singularity is almost isolated and of finite exponential order; (d) for a sequence of n of density unity if the singularity is virtually isolated. Furthermore, if the sole singularity of $f(z)$ on the circle of convergence is virtually isolated, then the density of the nonsmall coefficients c_n is unity.

M. S. Robertson.

Pólya, George. On converse gap theorems. Trans. Amer. Math. Soc. 52, 65–71 (1942). [MF 6994]

The author proves that two important gap theorems are best possible. Let $\{\lambda_n\}$, $n>0$, be an increasing sequence of positive integers. Then the author's first theorem is that a necessary condition that all series of the form (1) $a_1z^{\lambda_1}+a_2z^{\lambda_2}+\cdots$, with finite and nonzero radius of convergence, have their circle of convergence as a natural boundary is that (2) $\lim_{n\to\infty}n/\lambda_n=0$. This is the converse of the well-known Fabry gap theorem which tells us that (2) is a sufficient condition. The author also proves the converse of an interesting gap theorem of his own. His second theorem is that a necessary condition that all series of the form (1), with finite and nonzero radius of convergence, define uniform analytic functions is that $\liminf_{n\to\infty}n/\lambda_n=0$. The proofs of these theorems make use of results in the author's fundamental paper [Math. Z. **29**, 549–640 (1929)].

N. Levinson (Cambridge, Mass.).

Wall, H. S. The behavior of certain Stieltjes continued fractions near the singular line. Bull. Amer. Math. Soc. 48, 427–431 (1942). [MF 6708]

The author proves that, if $0<h_n<1$ $(n=1, 2, 3, \cdots)$ and if the series $\sum|h_n-\frac{1}{2}|$ converges, the function $f(z)$ represented by the continued fraction $K[a_n/1]$, where $a_1=h_1$, $a_n=(1-h_{n-1})h_nz$, approaches a finite continuous limit $\alpha(s)$ as $z\to-s(s\geqq1)$ from the upper half-plane, and approaches the complex conjugate of $\alpha(s)$ as $z\to-s$ from the lower half-plane. Further $f(z)$, which is known to be analytic in the complex plane after the negative real axis has been deleted, is proved to be bounded in absolute value in its domain of analyticity. The proof is based upon earlier investigations of the author [Trans. Amer. Math. Soc. 48, 165–184 (1940); Bull. Amer. Math. Soc. 47, 405–423 (1941); these Rev. 2, 90, 351] and upon a theorem due to I. Schur [J. Reine Angew. Math. 147, 205–232 (1917); 148, 122–145 (1918)]. *W. Leighton* (Houston, Tex.).

Chuang, Chi-Tai. Sur les fonctions holomorphes dans le cercle unité. Bull. Soc. Math. France 68, 11–40 (1940). [MF 6806]

The author proves the following theorem by a method due to Macintyre [Math. Z. 44, 536–540 (1938)] and applies it to majorize functions which are analytic for $|z|<1$ and are subject to various restrictions. Let $k(\geqq1)$ be an integer and let $g(x)$ be a real-valued function defined for $x\geqq0$ which is continuous and monotonic increasing for $x>X(g)$ $(\lim_{x\to+\infty} g(x)=+\infty)$ and satisfies

$$\lim_{n\to+\infty}\frac{g(n)}{n}=0,\quad \varliminf_{n\to+\infty}\frac{g(n)}{\log n}>ak^4,\quad n \text{ a whole number,}$$

where a is a positive constant. Let $g^{-1}(y)$ be the inverse of $g(x)$, the range of $g^{-1}(y)$ being restricted to $x>X(g)$. There exist positive numbers $\lambda(k, g)$, $\alpha(k)$, $\beta(k)$, $\beta'(k)$, $\beta_p(k)$ with the following properties. If $f(z)=\sum_{n=0}^{\infty}c_nz^n$ is analytic for $|z|<1$ and is not majorized by

$$\lambda(k, g)\left[S+\sum_{i=0}^{k-1}|c_i|\right]\sum_{n=0}^{\infty}e^{g(n)}z^n,$$

where S is an arbitrary given positive number, one can find in the annulus $\alpha(k)<|z|<1$ a circle $|z|=r$ $[r=r(k, g, f, S)]$ satisfying the condition $M(r, f)>S$, and k regions D_i $[D_i=D_i(r, f, \omega), i=1, 2, \cdots, k]$ crossed by the circle $|z|=r$ in which regions $f(z)$ behaves in the following manner: (1) $f(z)$ is univalent in D_i, mapping D_i on the slit annulus

$$\tfrac{1}{2}M(r, f)<|Z|<2M(r, f)|\arg Z-\omega|<\pi,\quad \omega \text{ arbitrary.}$$

(2) If $1\leqq m\leqq k$, the variation of $\arg f^{(m)}(z)$ in D_i is less than 3π and

$$\frac{1}{3}<\left|\frac{f^{(m)}(z)}{H_mf(z)}\right|<3,$$

$$\beta'(k)<H_m^{1/m}<\beta(k)\left[\log\frac{M(r, f)}{S}\right]^2g^{-1}\left[2\log\frac{M(r, f)}{S}\right],$$

H_m being a number depending on r, $f(z)$, k and m. (3) If $1\leqq m\leqq p$, then in D_i

$$|f^{(m)}(z)|<\beta_p(k)M(r, f)\left[\log\frac{M(r, f)}{S}\right]^{2p}g^{-1}\left[2\log\frac{M(r, f)}{S}\right]^p.$$

M. H. Heins (Chicago, Ill.).

Carathéodory, C. Über das Maximum des absoluten Betrages des Differenzenquotienten für unimodular beschränkte Funktionen. Math. Z. 47, 468–488 (1941). [MF 6729]

The author solves the following extremal problem [cf. Carathéodory, Bull. Amer. Math. Soc. 43, 231–241 (1937)]: Let $f(z)$ belong to the class K of functions which are analytic and of modulus less than unity for $|z|<1$ and which vanish for $z=0$. Further, let z_1 and z_2 be two given distinct points in the interior of the unit circle. To determine

$$M(z_1, z_2)=\operatorname*{l.u.b.}_{f\in K}\left|\frac{f(z_2)-f(z_1)}{z_2-z_1}\right|.$$

The problem is referred to the theory of cubic algebraic curves; it is demonstrated that

$$\begin{cases} M(z_1, z_2)=\cosh(\tau-\sigma) & \text{for } \tau>\sigma,\\ M(z_1, z_2)=1 & \text{for } \tau\leqq\sigma,\end{cases}$$

where σ and τ are the real, nonnegative solutions of

$$\cosh\sigma=\left|\frac{1-\bar{z}_1z_2}{z_2-z_1}\right|,\quad \cosh\tau=\frac{|z_1|+|z_2|}{|z_2-z_1|},$$

respectively. The properties of $M(h, z)$ (h real, $0<h<1$) and of its level curves are studied in detail. It is further shown that, if $|\arg z_2/z_1|=\theta$ $(0\leqq\theta<\pi)$, then $M(z_1, z_2)<1/\sin(\theta/2)$ and that this bound cannot be improved. *M. H. Heins* (Chicago, Ill.).

Hardy, G. H. and Littlewood, J. E. Theorems concerning mean values of analytic or harmonic functions. Quart. J. Math., Oxford Ser. 12, 221–256 (1941). [MF 6548]

Extracts from the introduction: "In this paper we complete (with one reservation stated at the end of this section) our account of a body of work which has occupied us at intervals since 1924. This work has been a piecemeal growth, and the logical order in which it stands is in some ways odd and anomalous. We can now give a more unified account of the main results. . . . The critical proofs will also be much shorter. We cannot claim without reservation that they are simpler, because we allow ourselves to appeal to certain theorems of Littlewood and Paley which were not available until recently; and the proofs of these theorems (of one in particular) are difficult. We could avoid appealing to these theorems if we chose. . . . In one part, where we thought we should be compelled to use them, we now find them unnecessary. . . . The paper is not a mere revision of old work: it contains proofs of theorems stated before without proof, and one entirely new theorem (theorem 8). In one respect, however, we do less than in our earlier papers, since we suppose throughout that the parameter r is greater than 1."

The theorems concern functions $f(z)$, $z=\rho e^{i\theta}$, regular for $\rho<1$; $\mathfrak{M}_r(f)=\mathfrak{M}_r(f, \rho)$ denotes the mean value

$$\left\{(2\pi)^{-1}\int_{-\pi}^{\pi}|f(\rho e^{i\theta})|^rd\theta\right\}^{1/r}.$$

The new theorem (theorem 8) states that, if $r>1$, $s>1$, $b<1$, and $f(0)=0$, then the quotient

$$\int_0^1(1-\rho)^{-bs}\mathfrak{M}_r^s(f)d\rho\Big/\int_0^1(1-\rho)^{s-bs}\mathfrak{M}_r^s(f')d\rho$$

is bounded above and below by positive numbers depending only on the parameters. The principal theorem (theorem 13)

deals with three functions $f(z)=\sum_1^\infty c_n z^n$, $g(z)=\sum_1^\infty b_n z^n$ and $h(z)=\sum_1^\infty b_n c_n z^n$; f and g satisfy $\mathfrak{M}_r(f)\leqq 1$, $\mathfrak{M}_\sigma(g')\leqq(1-\rho)^{k-1}$, with $\sigma\geqq 1$, $0<k<r^{-1}+\sigma^{-1}-1$; the conclusion is that $\mathfrak{M}_s(h)\leqq B$ if $s^{-1}=r^{-1}-k+\sigma^{-1}-1$, where B depends only on the parameters. This is a generalization of a theorem (theorem 12) about fractional integrals: if $r<s$, $\alpha=r^{-1}-s^{-1}$ and $f_\alpha(z)=\sum_1^\infty(in)^{-\alpha}c_n z^n$, then $\mathfrak{M}_s(f_\alpha)\leqq B\mathfrak{M}_r(f)$. The proof of theorem 13 now contains as a special case a "function-theoretic" proof of theorem 12, which was formerly derived from inequalities for certain bilinear forms, the proofs of which depend on theorems about rearrangements. The authors indicate how, conversely, the inequalities for bilinear forms could be deduced from theorem 12. Theorem 13 remains true with $k=0$ provided that $1\leqq r\leqq 2\leqq s$. It is shown, however, to fail if $r\leqq s<2$ or $2<r\leqq s$; it is in the proof of this fact that the results of Littlewood and Paley are no longer needed. It is also shown that theorem 13 fails if $k<0$. *R. P. Boas, Jr.* (Chapel Hill, N. C.).

Paatero, V. Über beschränkte Funktionen, welche gegebene Paare von Randbogen ineinander überführen. Math. Z. 47, 175–186 (1941). [MF 6817]

Löwner's lemma is concerned with analytic functions $w=f(z)$ which map the unit circle $|z|<1$ on a portion of the unit circle $|w|<1$, with $f(0)=0$, and which remain continuous on $|z|=1$ and map an arc α of $|z|=1$ on an arc β of $|w|=1$; the conclusion is that meas $\beta\geqq$meas α. This lemma gives rise to an existence problem when there are more than one pair of arcs involved. The author previously has solved the problem for two pairs of arcs; in the present paper the result is extended to three pairs. Necessary and sufficient conditions for the existence of mapping functions are given in terms of cross ratios of sets of the points involved. [For a direct extension of Löwner's lemma, also concerned with more than one pair of arcs, see the papers by Yosiro Kawakami, Jap. J. Math. 17, 569–572 (1941); these Rev. 3, 202.] *E. F. Beckenbach.*

Warschawski, S. E. On conformal mapping of infinite strips. Trans. Amer. Math. Soc. 51, 280–335 (1942). [MF 6318]

The author is concerned with a conformal mapping $Z(w)$ of an infinite strip in the $w=u+iv$ plane defined by $\varphi_-(u)<v<\varphi_+(u)$ $(-\infty<u<+\infty)$, where φ_- and φ_+ are continuous in u, onto the strip $|\Im z|<\pi/2$ in the z-plane, where the mapping is normalized to have the property $\lim_{u\to+\infty}\Re[Z(w)]=+\infty$. In particular, the conformal mapping of L-strips with boundary inclination γ at $u=+\infty$ $(|\gamma|<\pi/2)$ is considered, such strips being defined as infinite strips with the property that for $u_2>u_1$

$$\lim_{u_1\to+\infty}\frac{\varphi_+(u_2)-\varphi_+(u_1)}{u_2-u_1}=\lim_{u_1\to+\infty}\frac{\varphi_-(u_2)-\varphi_-(u_1)}{u_2-u_1}=\tan\gamma.$$

The main object of the paper is to obtain asymptotic expressions for $Z(w)$ and $Z'(w)$ as $u\to+\infty$.

To this end the author establishes the following two inequalities, related to those of Ahlfors [Acta Soc. Sci. Fenn. (N.S.) A. 1, no. 9 (1930)]: (I) If S is an L-strip with boundary inclination $\gamma=0$ at $u=+\infty$, then

$$\Re[Z(w_2)]-\Re[Z(w_1)]\leqq\pi\int_{u_1}^{u_2}\frac{1+\psi'^2(u)}{\theta(u)}du+\frac{\pi}{12}\int_{u_1}^{u_2}\frac{\theta'^2(u)}{\theta(u)}du+o(1),$$

where $o(1)\to 0$ as u_1, $u_2\to+\infty$, uniformly with respect to v_1 and v_2, and where

$$\theta(u)=\varphi_+(u)-\varphi_-(u),\quad \psi(u)=\tfrac{1}{2}[\varphi_+(u)+\varphi_-(u)].$$

(II) If S is an L-strip as in (I), and if $\varphi_+'(u)$, $\varphi_-'(u)$ are continuous and of bounded variation for $u_0\leqq u\leqq+\infty$, then

$$\Re[Z(w_2)]-\Re[Z(w_1)]\geqq\pi\int_{u_1}^{u_2}\frac{1+\psi'^2(u)}{\theta(u)}du-\frac{\pi}{4}\int_{u_1}^{u_2}\frac{\theta'^2(u)}{\theta(u)}du+o(1).$$

Under the hypothesis of (I) the following asymptotic representations are obtained:

$$\text{(A)}\qquad Z(w)=\lambda+\pi\int_{u_0}^{u}\frac{1+\psi'^2(t)}{\theta(t)}dt+i\pi\frac{v-\psi(u)}{\theta(u)}+o(1)$$

as $u\to+\infty$, uniformly with respect to v. Here λ is a real constant. (B) $Z'(w)\sim\pi/\theta(u)$ as $u\to+\infty$, uniformly in any subregion S_β:

$$\left\{\frac{|v-\psi(u)|}{\theta(u)}\leqq\frac{\beta}{\pi}\right\},$$

where $0<\beta<\pi/2$. Related asymptotic expressions are obtained for the case where $0<|\gamma|<\pi/2$. As an aid to deriving these results a theorem of Ostrowski pertaining to the argument of the derivative of the mapping function in the neighborhood of a cusp is employed [Acta Math. 64, 81–185 (1935)] a new proof of this theorem being given.

The results obtained are applied to various problems in the theory of the behavior of the mapping function in the neighborhood of the boundary under various hypotheses on the boundary (existence of a tangent, the presence of a cusp, two "concurrent" spirals, etc.). *M. H. Heins.*

Miser, Hugh J. Regions and their "patterns" in conformal mapping. Nat. Math. Mag. 16, 333–337 (1942). [MF 6483].

Let $w=f(z)$ $(f(0)=0)$ map $|z|<1$ (1, 1) and directly conformally onto a region S of the finite w-plane, and let S_r $(0<r<1)$ denote the image of $|z|<r$ under $f(z)$. The author studies the set $\Sigma(S)$ of complex numbers t which have the property that $w|tw$ carries S into itself. It is shown that $\Sigma(S)$ is closed, lies in $|t|\leqq 1$, and that $\lim_{r\to 0}\Sigma(S_r)$ is the circle $|t|\leqq 1$. This note is related to a paper of L. R. Ford [Duke Math. J. 1, 103–104 (1935)]. *M. H. Heins.*

Courant, Richard. The conformal mapping of Riemann surfaces not of genus zero. Univ. Nac. Tucumán. Revista A. 2, 141–149 (1941). [MF 6754]

The parallel slit theorem, according to which any Riemann domain of connectivity k and genus zero can be mapped conformally on a domain consisting of the whole plane except k parallel slits, previously has been generalized by the author [Math. Z. 3, 114–122 (1919)] to include Riemann domains of genus p; these domains can be mapped conformally on domains formed by cutting the plane along $2p$ pairs of parallel slits whose edges are suitably coordinated. The present paper extends these results to the case of non-orientable surfaces: it is shown that every closed Riemann domain, orientable or not, with characteristic number k can be mapped conformally on a slit domain of the same characteristic number. The proof given is topological in character; the author assumes as well-known the proof by the Dirichlet principle of the existence of a potential function on the surface. *E. F. Beckenbach.*

Seidel, W. and Walsh, J. L. On approximation by euclidean and non-euclidean translations of an analytic function. Bull. Amer. Math. Soc. 47, 916–920 (1941). [MF 5937]

G. D. Birkhoff proved [C. R. Acad. Sci. Paris 189, 473–475 (1929)] that there exists an entire function $F(z)$ such that to an arbitrary entire function $f(z)$ there corresponds a sequence $\alpha_1, \alpha_2, \cdots$ depending on $f(z)$ with the property

$$(1) \qquad \lim_{n\to\infty} F(z+\alpha_n)=f(z)$$

for all z, uniformly for z on any closed bounded set. The authors of the present paper extend this result in two directions. First, they show that the same representation (1) holds for a function $f(z)$ which is analytic in an arbitrary simply connected region R. Secondly, they prove a non-Euclidean analogue of this result: There exists a function $\Phi(z)$ analytic in the region $|z|<1$ such that, given an arbitrary function $f(z)$ analytic in a simply connected sub-region R, one has, for a suitably chosen sequence $\alpha_1, \alpha_2, \cdots$, the relation

$$\lim_{n\to\infty} \Phi\left(\frac{z+\alpha_n}{1+\bar{\alpha}_n z}\right)=f(z)$$

for z in R, uniformly on any closed set interior to R. The authors remark that in both results the region R may be replaced by a point set E provided the function $f(z)$ can be represented by a sequence of polynomials on E.

S. E. Warschawski (St. Louis, Mo.).

Lammel, Ernst. Zum Interpolationsproblem im Einheitskreise meromorpher Funktionen. I. Math. Z. 47, 132–140 (1940). [MF 6304]

It has been shown by S. Takenaka, F. Malmquist and J. L. Walsh that there is a unique series of the form

$$\frac{c_0}{1-\bar{a}_1 z}+\sum_{\nu=1}^{\infty}\frac{c_\nu}{1-\bar{a}_{\nu+1}z}\prod_{\mu=1}^{\nu}\frac{z-a_\mu}{1-\bar{a}_\mu z},$$

where $\{a_\mu\}$ is a sequence in $|z|<1$ having no limit points on $|z|=1$, which converges uniformly to $f(z)$, a function regular in $|z|<1$. The author has previously shown that this theorem remains true for series of the type

$$c_0+\sum_{\nu=1}^{\infty} c_\nu \prod_{\mu=1}^{\nu}\frac{z-a_\mu}{1-\bar{a}_\mu z}.$$

These theorems are employed in solving the interpolation problem for functions regular in $|z|<1$ when no limit point of the interpolation values lies on the unit circle. Walsh has treated an analogous problem for functions meromorphic in a finite part of the plane. Let $\{b_\mu\}$ be a sequence in $|z|<1$ all of whose limit points lie on $|z|=1$, and let $H(b)$ designate the class of functions meromorphic in the unit circle whose poles belong to $\{b_\mu\}$. To each function of the class $H(b)$ there is a unique sequence $S_n(z)$ of rational functions, the partial sums of

$$(1) \qquad c_0+\sum_{\nu=1}^{\infty} c_\nu \prod_{\mu=1}^{\nu} z\left(\frac{z-b_\mu}{1-\bar{b}_\mu z}\right)^{-1},$$

with the property that $S_n^{(p)}(0)=f^{(p)}(0)$ $(p=0, 1, 2, \cdots)$, p indicating the order of differentiation. The author shows that a series of type (1) represents a function of the class $H(b)$ when and only when

$$(2) \qquad \overline{\lim_{\nu\to\infty}}\, c_\nu^{1/\nu}\leqq 1;$$

and conversely, each function $f(z)$ of the class $H(b)$ can be developed in a series of type (1) whose coefficients $\{c_\nu\}$ satisfy the relation (2). The author also finds the necessary and sufficient condition for a function whose poles belong to a given sequence $\{b_\mu\}$ all of whose limit points lie on $|z|=1$, and whose value, and those of its derivatives, at the origin have the respective values $\{z^{(\nu)}\}$, $(\nu=0, 1, \cdots)$. When such a function exists it can be represented uniquely by a series of form (1). *A. Gelbart* (Hampton, Va.).

Lammel, Ernst. Über Approximation meromorpher Funktionen durch rationale Funktionen. Math. Ann. 118, 134–144 (1941). [MF 6314]

In this paper the author obtains results for meromorphic functions similar to those he obtained for regular functions, and by similar methods [Monatsh. Math. Phys. 49, 199–208 (1940); these Rev. 2, 80]. Given a sequence of points $\{a_\mu\}$ $(\mu=1, 2, \cdots)$ in $|z|\leqq\rho<R$ and a sequence of points in $|z|<R$ all of whose limit points lie on $|z|=R$. If $f(z)$ is a meromorphic function in $|z|<R$, all of whose poles belong to $\{b_\mu\}$, then there is a unique sequence of rational functions $S_n(z)$, the partial sums of

$$(1) \qquad c_0+\sum_{\nu=1}^{\infty} c_\nu \prod_{\mu=1}^{\nu}\frac{z-a_\mu}{z-b_\mu},$$

having the properties that when a_λ occurs x times in the sequence $\{a_\mu\}$, then $S_n^{(p)}(a_\lambda)=f^{(p)}(a_\lambda)$ $(p=0,1,2,\cdots,x-1)$, $f^{(0)}(z)=f(z)$. The author shows that the necessary and sufficient relationship of the sequences $\{a_\mu\}$ and $\{b_\mu\}$ in order that the sequence of partial sums S_n converges to $f(z)$ for all points of $|z|<R$, except when z belongs to $\{b_\mu\}$, is

$$(2) \qquad \lim n^{-1}\sum_{\mu=1}^{n}(a_\mu{}^k-b_\mu{}^k)=0, \qquad k=1, 2, \cdots.$$

The author also considers an arbitrary bounded open region B whose boundary is a closed Jordan curve C, and shows that, in order for the rational functions $S_n(z)$ to converge to $f(z)$ for all values of z in B, except for those belonging to $\{b_\mu\}$, the two sequences into which $\{a_\mu\}$ and $\{b_\mu\}$ are mapped by the function $z=z(\zeta)$ mapping B into $|\zeta|<1$ must be so related as to satisfy (2). *A. Gelbart.*

Botella Raduán, F. The group admitted by functions of a complex variable and Riemann surfaces and its relation to corresponding Riemann spaces of vanishing curvature. Revista Mat. Hisp.-Amer. (4) 2, 22–32 (1942). (Spanish) [MF 7082]

The author has noticed that, if $f(z)$ is analytic, both $|f(z)dz|^2$ and $|f'(z)dz|^2$ define on the Riemann surface of f locally Euclidean Riemannian metrics, the latter of which is invariant under any group which leaves f invariant. This remark is amplified in various ways. *A. Weil.*

Wachs, S. Sur quelques propriétés des transformations pseudo-conformes avec un point frontière invariant. Bull. Soc. Math. France 68, 177–198 (1940). [MF 6814]

The theorem of Julia on the transformation of the interior of a domain with a fixed-point on the boundary can be deduced from the Riemann mapping theorem and the Schwarz-Pick lemma. Bergman has obtained certain analogues of these theorems for mapping by functions of two variables. Using the so-called kernel function, he defines a Hermitian metric which is invariant with respect to pseudo-conformal transformations. Bergman shows that the

neighborhood of a given boundary point maps into a standard domain. With these results and the theorem (of Bergman) that for the hypersphere the non-Euclidean length does not decrease in an interior transformation, the author proves the following: Let $Q(0,0)$ be a boundary point of B of the third order (that is, the standard domain being a hypersphere) and let $W=[w_k=w_k(z_1, z_2), k=1, 2]$ be a pseudo-conformal transformation mapping B into $G\subset B$, with the fixed-point at $(0, 0)$. If in the neighborhood of a point Q a sequence $\{z^{(n)}\}$, $\lim_{n\to\infty} z^{(n)}=Q$, exists such that

$$\lim_{n\to\infty} F(w_1^{*(n)}, w_2^{*(n)})/F(z_1^{*(n)}, z_2^{*(n)})=\Gamma_1, \quad 0<\Gamma_1<\infty,$$

then $\Gamma_1B(w_1^*, w_2^*)\geqq B(z_1'^*, z_2'^*)$. Here $F(u_1, u_2)=\rho_1(u_1+\bar{u}_1)-|u_1|^2-|u_2|^2$, $B(u_1, u_2)=F/|u_1|^2$, $w_1^*=w_1(1+\alpha_2w_1)^{-1}$, $w_2^*=w_2(1+\beta_2w)^{-1}$, $z_1^*=\rho_2z_1(\rho_1-\alpha_1\rho_1z_1)^{-1}$, $z_2'^*=\rho_2/\rho_1z_2\times(1+\beta_1z_1)^{-1}$, where α, β and ρ are constants depending only upon B. Here z_1 and z_2 are chosen in such a way that x_1 coincides with the interior normal, and $x_1=0$ is the tangential hyperplane at the point Q. Under the further condition that $\mathfrak{H}=[|z_1-\rho|^2+|z_2|^2-\rho^2\leqq 0]$ lies in G and the condition that a squence of points exist such that

$$\lim_{n\to\infty}F(\eta_1^{(n)}, \eta_2^{(n)})/F(z_1'^{(n)}, z_2'^{(n)})=\Gamma_2$$

exists, where $z'_k=z_k/(1+\gamma_kz_k)$, and $\eta_k=\rho w_k/\rho_1$, the author establishes the inequality $\Gamma_2B(\eta_1, \eta_2)\leqq B(z_1', z_2')$. Finally under all the above conditions the author proves that, in a certain region (cone) about Q, $0<A\leqq|\partial(z_1, z_2)/\partial(w_1, w_2)|\leqq B<\infty$. *A. Gelbart* (Hampton, Va.).

Functional Analysis, Ergodic Theory

***Halmos, Paul R. Finite Dimensional Vector Spaces.** Annals of Mathematics Studies, no. 7. Princeton University Press, Princeton, N. J., 1942. v + 196 pp. $2.35.

A purpose of this study is to present n-dimensional transformation theory from the abstract point of view. Normally, due to the existence of a finite basis, elementary matrix theory has an aspect not at all suggesting that the infinite dimensional theory of operators is a natural extension of it. The author exploits as completely as possible the methods and notions of the infinite in his presentation of the finite case; such a program has long been needed. Thus we find transformations studied with the help of adjoint spaces, reflexiveness, linear manifolds, disjointness, projections, etc. The book is written with great care; it should be much appreciated by the young graduate student who wishes to begin his studies of linear mathematics.

In developing the subject abstract vector spaces are first defined; the norm is not mentioned, its place being taken by the finite dimensionality of the spaces. This property makes them reflexive, thereby endowing them with very useful qualities. The notion of basis is developed. When linear transformations are introduced, it is shown how any basis generates a matrix. A complete interpretation of transformation problems in the matrix terminology is given. Probably the two most important problems of linear transformation theory are: (1) to determine the structure of symmetric (normal) transformations in spaces possessing an inner product, that is, unitary or complex Euclidean spaces; (2) to determine the structure of an arbitrary linear transformation in an arbitrary space. The canonical forms are the real diagonal and the Jordan form, respectively. The first problem is treated in the book's most important chapter; the second is relegated to a rather brief appendix. This seems unfortunate. Since the principal unsolved and not always correctly understood problems lie in the direction of the latter, a considerable service would have been performed had the Jordan form been accorded the same unhurried, well-motivated and thorough treatment one finds throughout earlier pages. One feels inclined to predict that the general case of problem two will receive the most attention in linear developments of the near future.

E. R. Lorch (New York, N. Y.).

Pospíšil, Bedřich. Von den Verteilungen auf Booleschen Ringen. Math. Ann. 118, 32–40 (1941). [MF 6308]

The present paper is a sequel to the author's "Über die messbaren Funktionen" [Math. Ann. 117, 327–355 (1940); these Rev. 2, 131]. He first gives four equivalent conditions for a real-valued function $h(\varphi)$ defined on a Boolean ring A of elements φ to be "continuous." He defines A to be "separable" when, effectively, its representation by open-and-closed subsets of a totally disconnected bicompact Hausdorff space [M. H. Stone] is separable. He shows that separability is necessary and sufficient for there to exist an isomorphism between the set Φ of all bounded continuous functions from finite sets of intervals of reals to A, and a subring of the ring c of bounded sequences of real numbers. Various other results are given. *G. Birkhoff.*

Olmsted, John M. H. Lebesgue theory on a Boolean algebra. Trans. Amer. Math. Soc. 51, 164–193 (1942). [MF 6093]

The author first constructs from any σ-complete Boolean algebra B a σ-complete vector lattice $\Omega(B)$ with weak unit. He shows that it contains every other σ-complete vector lattice F having B for the Boolean algebra of its complemented 1-ideals (normal subspaces); the construction of B from F was accomplished by Freudenthal [Nederl. Akad. Wetensch., Proc. 39, 641–651 (1936)]. In case B is the algebra of all subsets of a class I, $\Omega(B)$ is the vector lattice of all real functions on I.

The bounded functions on B form a complete (not merely σ-complete) vector sublattice of $\Omega(B)$. If B has a continuous finite measure, then we can define L^p-spaces on B; the author characterizes such spaces axiomatically [cf. also H. F. Bohnenblust, Duke Math. J. 6, 627–640 (1940); these Rev. 2, 102 for the separable case]. If B has a continuous finite measure, one can also construct the space of all absolutely continuous functions on B, and prove that every additive, nonnegative, absolutely continuous function is the "integral" of its "derivative." *G. Birkhoff.*

Maharam, Dorothy. On measure in abstract sets. Trans. Amer. Math. Soc. 51, 413–433 (1942). [MF 6323]

The author shows that a postulated class Φ of σ-isomorphisms on the principal ideals of a Boolean σ-algebra M gives rise to an equivalence relation between elements of M which is countably additive and hereditary in the sense that, if $a\sim b$ and $a'<a$, then there is a $b'<b$ such that $a'\sim b'$. This equivalence relation is preserved under subtraction and taking limits of monotone sequences provided the elements involved are bounded, that is, not equivalent to any proper subelement. The measure of an element of M is taken to be the class of all its equivalent elements and the set $\mathfrak{M}$ of measures of bounded elements satisfies conditions that are generalizations of the Lebesgue conditions.

A complete measure is also discussed, and the Carathéodory condition for measurability is proved. Conditions are given under which the set $\mathfrak{M}$ is isomorphic to a set of nonnegative numbers, the isomorphism being unique up to a multiplicative constant. In the last part of the paper the author considers a Boolean σ-algebra M upon which a numerical measure is defined and gives conditions sufficient that sets of equal measure should be equivalent under some σ-isomorphism and that the given measure be obtainable as described above. A generalization of a theorem by Banach and Tarski [Fund. Math. 6, 244–277 (1924), in particular, p. 277] is also obtained. *H. H. Goldstine.*

Maharam, Dorothy. On homogeneous measure algebras. Proc. Nat. Acad. Sci. U.S.A. **28**, 108–111 (1942). [MF 6333]

In another paper [see the preceding review] the author considered a Boolean σ-algebra $M=[0\leqq a\leqq b\leqq\cdots\leqq e]$ in which was defined a numerical measure. Such a measure algebra is called homogeneous if the power of every principal ideal (other than the null ideal) is equal to the power of a σ-basis of M. These algebras are characterized in this paper. Consider the Boolean algebra $P(\gamma)$ of all measurable sets (mod null sets) of the product space $P_{0\leqq\alpha<\gamma}I_\alpha$ of intervals $0\leqq x_\alpha\leqq 1$, where α, γ are ordinals. An elementary set $E=P_{0\leqq\alpha<\gamma}A_\alpha$ in this space is one for which only a finite number of A_α are different from I_α and the measure of E is $\prod mA_\alpha$. The measurable sets are then built up from the elementary ones. It is proved that every homogeneous measure algebra with $me=1$ is isomorphic to $P(\gamma_0)$, where γ_0 is the least ordinal corresponding to M, that is, there is a measure-preserving σ-isomorphism between them. It is, moreover, shown that every measure algebra is a direct sum of homogeneous algebras. Let M be a measure algebra, and let G be the group of all measure preserving σ-automorphisms of M. The author proves that M is a direct sum of a denumerable number of invariant principal ideals on each of which G is ergodic *H. H. Goldstine.*

Randall, Merle and Longtin, Bruce. Intuitive and descriptive geometry of function space: The graphical representation of geometrical figures. J. Washington Acad. Sci. **31**, 421–431 (1941). [MF 6366]

Longtin, Bruce and Randall, Merle. Intuitive and descriptive geometry of function space: Metric properties and transformation of coordinates. J. Washington Acad. Sci. **31**, 441–453 (1941). [MF 6367]

Randall, Merle and Longtin, Bruce. Intuitive and descriptive geometry of function space: Geometric configurations. J. Washington Acad. Sci. **31**, 453–466 (1941). [MF 6368]

Longtin, Bruce and Randall, Merle. Intuitive and descriptive geometry of function space: Tensors and bivectors. J. Washington Acad. Sci. **31**, 485–495 (1941). [MF 6369]

These four papers, written by two chemists, deal with different devices for increasing one's geometrical intuition of situations in function spaces and in Euclidean spaces of high dimensionality. Various methods of graphical representation of configurations and operations in such spaces are described. One method illustrated consists of projecting the function space configuration in different ways onto a two- or three-dimensional space. Another involves projecting onto an n-dimensional space, and then representing a point in the latter by means of a set of n ordinates erected at equally spaced points of an interval, or by means of a broken line joining extremities of these ordinates. Many figures and diagrams are included by way of illustration. Graphical representations of infinite matrices and of points with complex coordinates are also described. The second paper contains a collection of formulas dealing with orthonormal bases and transformations of coordinates in Hilbert space. There are numerous inaccuracies in the mathematical statements in these papers. *O. Frink.*

Taylor, Angus E. The weak topologies of Banach spaces. Revista Ci., Lima **42**, 355–366 (1940); **43**, 465–474, 667–674 (1941); **44**, 45–63 (1942). [MF 2899]

The author is concerned with various topologies in Banach spaces and their relationship with completeness, continuity and the property of being reflexive. Being given a linear topological space, the author also considers, in addition to the topological completeness as defined by von Neumann, completeness with respect to all directed sets and "bounded completeness" (implying the existence of a limit for every bounded fundamental directed set). The author makes progress (but does not solve completely) in the problem of extending a given linear topological space to a boundedly complete space. One of the most interesting results concerning continuity is a theorem (5.2) to the effect that, if T is a distributive operation on a Banach space E_1 to another Banach space E_2 and if T is continuous in certain weak topologies of E_1, E_2, then T will be continuous in the metric topologies of E_1, E_2. Concerning reflexiveness the author proves that the Banach space E is reflexive if and only if it is boundedly complete. The lack of space prevents a more detailed analysis of the paper. *J. D. Tamarkin.*

Smiley, M. F. A remark on S. Kakutani's characterization of (L)-spaces. Ann. of Math. (2) **43**, 528–529 (1942). [MF 7007]

The author shows that axiom (V) in Kakutani's paper "Concrete representation of abstract (L)-spaces, etc.," [Ann. of Math. (2) **42**, 523–537 (1941); these Rev. **2**, 318] is redundant. *G. Birkhoff* (Cambridge, Mass.).

Mackey, George W. Isomorphisms of normed linear spaces. Ann. of Math. (2) **43**, 244–260 (1942). [MF 6461]

Let X_1 and X_2 be normed linear spaces (not necessarily complete). Let L_i ($i=1, 2$) be the lattice of closed linear subspaces of X_i. The first main theorem is that X_1 and X_2 are isomorphic, in the sense of Banach, if and only if L_1 and L_2 are isomorphic as lattices. The second theorem states that, if R_i is the ring of all continuous linear transformations of X_i into itself, X_1 and X_2 are isomorphic if and only if R_1 and R_2 are isomorphic as rings. For the case that X_1 and X_2 are complete, this theorem is due to Eidelheit [Studia Math. **9**, 97–105 (1940); these Rev. **3**, 51]. Finally, if G_i is the group of all 1-1 bicontinuous linear transformations of X_i into all of itself, it is proved that G_1 and G_2 are isomorphic as groups if and only if either (a) X_1 and X_2 are isomorphic or (b) they are pseudo-reflexive and mutually pseudo-conjugate. The concept of pseudo-reflexivity (too involved to be presented in a brief review) is such that a pseudo-reflexive space is reflexive if and only if it is complete.

The proofs of all three theorems rest basically upon two lemmas. The first of these is a generalization of the theorem that a collineation between two real projective planes is effected by a linear transformation. The second lemma states that, if X_1 and X_2 are placed in 1-1 correspondence by a

linear transformation T, then T is a homeomorphism provided that T and T^{-1} leave intact the property of being a maximal closed subspace. *A. E. Taylor.*

Snapper, Ernst. Structure of linear sets. Trans. Amer. Math. Soc. **52**, 257–264 (1942). [MF 7120]

The author extends to finite dimensional vector spaces with scalars out of an integral domain the Noether theory on the decomposition of ideals into primary ideals. The vectors of the space V_n are n-tuples from the scalar domain R. Every ideal in R is assumed to have a finite basis. If the dimension n is 1, the present theory reduces immediately to the classic case. The analogues of ideals in V_n are linear sets, that is, subsets of V_n closed under vector subtraction and scalar multiplication. For the most part, the extension is straightforward, paralleling closely the development given in van der Waerden [Moderne Algebra, vol. 2, Springer, Berlin, 1931, chap. 12]. However, some new notions, trivial in R but not in V_n, enter, for example, those of closure, density and essential ideal of a linear set L. The essential ideal E of L is defined by $E=L/V_n$. A set L is primary if $\lambda v\equiv 0(L)$, $\lambda\epsilon R$, $v\epsilon V_n$, implies either $v\equiv 0(L)$ or $\lambda\equiv 0(E')$, where E' is the radical of E. The first decomposition theorem is: Any linear set is the intersection of a finite number of primary linear sets. *E. R. Lorch* (New York, N. Y.).

Tukey, J. W. Some notes on the separation of convex sets. Portugaliae Math. **3**, 95–102 (1942). [MF 6669]

Disjoint sets A, B in a normed vector space are said to be separated by a plane if there exists a linear functional f, not identically zero, and a real number c such that $f(x)\geqq c$, $x\epsilon A$ and $f(x)\leqq c$, $x\epsilon B$. If A and B are disjoint convex sets then they can be separated by a plane under each of the following conditions: (1) A is open; (2) A is closed and sequentially (weakly sequentially or weakly) compact and B is closed; (3) the space is reflexive, for example, Hilbert space, A is closed and bounded and B is closed. These results follow from a theorem of Mazur [Studia Math. 4, 70–84 (1933), in particular p. 73] and a lemma stating that A and B are separated by f if and only if the set of $a-b$, $a\epsilon A$, $b\epsilon B$, is separated by f from Θ. Examples are given showing the necessity of boundedness in (3). Finally it is shown that an infinite dimensional normed vector space is the sum of two complementary dense convex sets.

L. W. Cohen (Lexington, Ky.).

Van der Lijn, Gaston. Les polynomes abstraits (Suite II). Bull. Sci. Math. (2) **64**, 128–144 (1940). [MF 6787]

This is part III of the author's treatise on abstract polynomials which has appeared serially [Bull. Sci. Math. (2) **64**, 55–80, 102–112, 163–196 (1940); cf. these Rev. **1**, 259; **2**, 222, 419; **3**, 50]. Here the notion of continuity for abstract polynomials on a normed space is introduced and a number of known results are obtained. *F. J. Murray.*

Tseng, Y. Y. On generalized biorthogonal expansions in metric and unitary spaces. Proc. Nat. Acad. Sci. U.S.A. **28**, 170–175 (1942). [MF 6604]

The author announced a number of results connecting expansion properties of a set $\{f_p\}$ of elements of a complete metric space with those of another set $\{g_p\}$. In particular, he obtains results for more general sets, and sharper inequalities, than appear in previous work of similar character by Paley and Wiener [Fourier Transforms in the Complex Domain, American Mathematical Society, New York, 1934, pp. 100 ff.], Boas [Trans. Amer. Math. Soc. **48**, 467–487 (1940); these Rev. **2**, 80], and Duffin and Eachus [abstract in Bull. Amer. Math. Soc. **46**, 415 (1940)]. The results are expressed in the language of E. H. Moore's general analysis; the methods, using results from general analysis, appear to be of different character from those previously used in problems of this kind. *R. P. Boas, Jr.*

Cramér, Harald. On harmonic analysis in certain functional spaces. Ark. Mat. Astr. Fys. **28B**, no. 12, 7 pp. (1942). [MF 6953]

The author proves the following theorem. If dP is a measure of a general type on the space S of finite functions $f\equiv f(t)=g(t)+ih(t)$, $-\infty<t<\infty$, if the measure is invariant under translations on the t-axis, if the function $\phi(t)=\int_S f(t_0+t)\overline{f(t_0)}dP$ is continuous and if $\Phi(x)$, $-\infty<x<\infty$, is defined by $\phi(t)=\int_{-\infty}^{\infty}e^{itx}d\Phi(x)$, then there exists a linear transformation $F(x)=T(f(t))$ such that (i) for any continuity points $a<b$ of $\Phi(x)$ the relation

$$\Phi(b)-\Phi(a)\cdot\int_S|F(b)-F(a)|^2dP$$

holds, and (ii) in suitably generalized Stieltjes integrals the relation $f(t)=\int_{-\infty}^{\infty}e^{itx}dF(x)$ holds for almost all f in S, the exceptional null set depending on the generalization used. This theorem is an "almost-all" counterpart to generalized harmonic analysis [N. Wiener, S. Bochner], and the author notes that the "almost-all" problem has been approached before by Wiener, Khintchine, Slutsky, etc. *S. Bochner.*

Hille, Einar. Representation of one-parameter semigroups of linear transformations. Proc. Nat. Acad. Sci. U.S.A. **28**, 175–178 (1942). [MF 6605]

If T_s, $0\leqq s$, is a weakly measurable semigroup of linear operators in a separable complex Banach space E satisfying the conditions $|T_s|\leqq 1$, $T_0=I$, and $T_s(E)$ is dense in E, then T_sx is strongly differentiable at $s=0$ to a closed linear transformation A whose domain $D(A)$ is dense in E. Several representations for T_s are stated; for example, if $s>0$ and $x\epsilon D(A)$ then for any $c>0$

$$T_sx=-\lim_{\omega\to\infty}(2\pi i)^{-1}\int_{c-i\omega}^{c+i\omega}c^{s\lambda}R(\lambda)d\lambda,$$

where $R(\lambda)$ is the resolvent of A and is given by the formula

$$R(\lambda)x=-\int_0^{\infty}e^{-\lambda s}T_sxds.$$

The results stated constitute an extension of similar results for groups due to M. H. Stone [Ann. of Math. (2) **33**, 643–648 (1932)], I. Gelfand [C. R. (Doklady) Acad. Sci. URSS (N.S.) **25**, 713–718 (1939); these Rev. **1**, 338] and M. Fukamiya [Proc. Imp. Acad. Tokyo **16**, 262–265 (1940); these Rev. **2**, 105]. *N. Dunford* (New Haven, Conn.).

Durañona y Vedia, Agustin. On linear operators in Hilbert space. Univ. Nac La Plata. Publ. Fac. Ci. Fisicomat. Serie 2: Revista **2**, 87–123 (1941). (Spanish) [MF 7132]

A resumé of a seminar on linear transformations which are either completely continuous, Schmidt class or self-adjoint. The results are known. *F. J. Murray.*

Nakano, Hidegorô. Unitärinvarianten im allgemeinen Euklidischen Raum. Math. Ann. **118**, 112–133 (1941). [MF 6313]

This paper presents a complete unitary invariant of a normal operator on a not necessarily separable unitary space. This problem was originally solved by F. Wecken

[Math. Ann. 116, 422–455 (1939)]. Let U be a set of operators, f an element of the space, $\mathfrak{M}(U, f)$ the closed linear manifold determined by the elements Rf for R in U. The essential difficulty of the problem is that the notion of multiplicity of the spectrum can no longer be expressed in terms of $\mathfrak{M}(\mathfrak{U}, f)$, where $\mathfrak{U}$ is the set of projections associated with the spectral resolution of a normal operator. While Wecken uses various notions involving monotonic functions to define multiplicity, Nakano defines the "dimensionality" of a set of commuting projections $\mathfrak{U}$ in the following way. Let the projection R commute with $\mathfrak{U}$. Then $R\mathfrak{U}$ is said to be isomorphic to $\mathfrak{U}$ if $RE_1 = RE_2E_1$ and E_2 in $\mathfrak{U}$ implies $E_1 = E_2$. If we have $R_1, \cdots, R_\alpha$ mutually perpendicular projections with $R_i\mathfrak{U}$ isomorphic to $\mathfrak{U}$, then the "dimensionality" of $\mathfrak{U}$ is at least α. The l.u.b. of such α's is the minimal "dimensionality" of $\mathfrak{U}$. Now in $\mathfrak{U}$ we can find mutually perpendicular E's such that $E_i\mathfrak{U}$ is of uniform multiplicity, that is, no subpart is of higher minimal dimensionality. If $\mathfrak{U}$ is the set of projections associated with a resolution of the identity, each $E_i\mathfrak{U}$ can then be characterized in terms of ideals of Borel sets. These last behave like the intersection of a noncountable number of Borel sets. A set of projections is said to be "fundamental" if it is determined by majorizing and intersecting the subsets of a countable number of commuting projections. A fundamental set $\mathfrak{U}$ may contain a noncountable number of mutually orthogonal projections but $\mathfrak{M}(\mathfrak{U}, f)$ is separable for every f. *F. J. Murray* (New York, N. Y.).

Day, Mahlon M. Operation in Banach spaces. Trans. Amer. Math. Soc. 51, 583–608 (1942). [MF 6554]

It is well known that the necessity of the boundedness condition for the regularity of a matrix follows from the theorem of Banach which asserts that the sequence of norms $|U_n|$ of the linear operators U_n is bounded provided $|U_na|$ is bounded for each a in the space upon which U_n operates. In generalizing the Toeplitz theorem on the regularity of a matrix by replacing the sequence U_n with a directed set U_x of operators, the chief problem becomes that of finding when $\limsup_x |U_xa| < \infty$ implies $\limsup_x |U_x| < \infty$. The author takes a first step towards a complete solution of this problem by giving, among other results, the following: (1) the range space B of the operators U_x on A to B is immaterial; (2) the desired implication is valid for any directed set X if and only if the domain space A is finite dimensional; and (3) examples of directed sets X are given, which can be defined for any A (not finite dimensional) such that the implication is false. An integral $\int f d\Phi$ is defined for an operator valued measure Φ and vector valued f and used to represent a certain type of linear operator. *N. Dunford.*

Day, Mahlon, M. Ergodic theorems for Abelian semigroups. Trans. Amer. Math. Soc. 51, 399–412 (1942). [MF 6322]

The method of F. Riesz [J. London Math. Soc. 13, 274–278 (1938)] is adapted to a considerably more general situation. The cyclic group of operators is replaced by a general semigroup T^s of operators in a Banach space B, and the means are replaced by an integral with respect to a directed set of nearly invariant additive set functions. This situation is quite analogous to that considered by Alaoglu and G. Birkhoff [Ann. of Math. (2) 41, 293–309 (1940); these Rev. 1, 339]. The chief differences are that Day uses a more general integral allowing him to dispense with measurability restrictions and that he shows that the directed set of the means of the transforms of any $b \epsilon B$ always has at least one limit point (in the weak * topology) when regarded as a directed set in B^{**}. This limit point is unique and in B if and only if b lies in the linear manifold determined by the fixed points of the semigroup and the vectors of the form $b - T^sb$. This linear manifold will be the whole of B providing the directed set has a countable cofinal subset and the unit sphere in B is sequentially weakly compact. *N. Dunford* (New Haven, Conn.).

Kakutani, Shizuo. Representation of measurable flows in Euclidean 3-space. Proc. Nat. Acad. Sci. U.S.A. 28, 16–21 (1942). [MF 6006]

Let Ω be a space with a Borel field $\mathfrak{B}$ of subsets and a finite measure m on $\mathfrak{B}$. Suppose $\Omega(\mathfrak{B}, m)$ is properly separable in the sense that there is a countable collection $\mathfrak{A}$ of sets in $\mathfrak{B}$ such that for every $M \epsilon \mathfrak{B}$ and for every $\epsilon > 0$ there is a subsequence $\{M_n\} \subset \mathfrak{A}$ such that $M \subset \sum_n M_n$, $\sum_n m(M_n) < m(M) + \epsilon$. Suppose also that there is a countable collection of subsets of Ω such that for every two distinct points of Ω there is at least one set in the collection which contains one but not both of the points. Finally let every point of Ω have measure zero. Then every measurable flow in $\Omega(\mathfrak{B}, m)$ is isomorphic to a continuous flow in a space $\Omega(\mathfrak{L}^*(\Omega), \mu^*)$, where Ω is a set in Euclidean 3-space with finite outer Lebesgue measure $\mu^*(\Omega)$ and $\mathfrak{L}^*(\Omega)$ is the field of sets of the form $\Omega \cdot \Lambda$, where Λ is Lebesgue measurable. Also every measurable flow defined on a Lebesgue measure space (of finite measure) in Euclidean n-space ($n \geqq 1$) is isomorphic to a continuous flow on a Lebesgue measure space in Euclidean 3-space. *N. Dunford* (New Haven, Conn.).

Siegel, Carl Ludwig. Some remarks concerning the stability of analytic mappings. Univ. Nac. Tucumán. Revista A. 2, 151–157 (1941). [MF 6755]

There are four remarks concerning transformations of two-dimensional space in the neighborhood of an invariant point. Remark 1 is purely expository giving a simple proof of known relationships between stability (or instability) and the nature of the characteristic roots of the transformation. Remark 2 points out a limitation in a well-known criterion of Levi-Civita for instability in certain cases when the characteristic roots have absolute value one. In remark 3, the author states a long since conjectured theorem to the effect that the well-known canonical form of Birkhoff in the so-called conservative formally stable case can not always be obtained by using convergent series. The proof (which is omitted) is said to be closely analogous to the proof given by the author of a similar well-known theorem in the theory of canonical differential equations. Remark 4 gives an example of an unstable algebraic area preserving transformation, one of whose characteristic roots λ is a preassigned root of unity (the other root is necessarily λ^{-1}). *D. C. Lewis* (Durham, N. H.).

Halmos, Paul R. and von Neumann, John. Operator methods in classical mechanics. II. Ann. of Math. (2) 43, 332–350 (1942). [MF 6465]

This paper continues earlier researches on measure preserving transformations by von Neumann [Ann. of Math. (2) 33, 587–642 (1932)]. Two measure spaces, that is, spaces on certain sets of which measures are defined, are called point isomorphic if one can be transformed into the other in a 1-1 measure preserving way (neglecting a set of measure 0 in each). Two measure spaces are called set iso-

morphic if there is a 1-1 measure preserving transformation of the measurable sets of one into those of the other (sets of measure 0 disregarded throughout) which takes sums into sums and complements into complements. Throughout the following, "conditions of type F" will mean conditions on the fields of measurable sets of the measure spaces involved. (1) Necessary and sufficient conditions of type F are found that a measure space be point isomorphic to the unit interval (with Lebesgue measure). (2) Under hypotheses of type F, necessary and sufficient conditions are found that every set automorphism of a measure space on itself be generated by a point transformation. (3) Under hypotheses of type F, it is shown that an ergodic measure preserving transformation T, with pure point spectrum, of a measure space into itself is point isomorphic to a rotation $x\to ax$ on a compact separable Abelian group (on which measure is Haar measure). This result makes possible simple proofs of properties of measure preserving transformations. Thus it is shown that T is necessarily isomorphic to its own inverse. (4) For the group rotations of (3), metric transitivity is equivalent to regional transitivity. (5) If T is an ergodic measure preserving transformation on a metric measure space X, with the usual relations between measure and metric, and if T is isometric, or more generally, if the family $\{T^n\}$ is equicontinuous, then T has a pure point spectrum; in fact, a multiplication can be defined on X so that X becomes a compact separable Abelian group, and T becomes a rotation. *J. L. Doob* (Washington, D.C.).

Wiener, Norbert and Wintner, Aurel. On the ergodic dynamics of almost periodic systems. Amer. J. Math. 63, 794–824 (1941). [MF 5627]

Let T^t, $-\infty<t<\infty$, denote a measure-preserving flow in a space S of finite Lebesgue measure. Under what circumstances can the function $F(t)=f(T^tP)$, where P is any point of S and $f(P)$ is a function of class $L^q(q\geqq1)$, be approximated in the qth mean by a sequence of trigonometric polynomials $r_n(x)$:

$$M_t\{|f(t)-r_n(t)|^q\} = \lim_{-A,\,B\to\infty}[1/(B-A)]\int_A^B |f(t)-r_n(t)|^q dt\to 0$$

as $n\to\infty$? In other words, when is the function $f(T^tP)$ almost periodic in the sense (B^q) of Besicovitch, except that A and B are independent rather than $B=A$? As the authors point out, this question is an important and natural one, suggested on the one hand by the divergent trigonometric developments of celestial mechanics, and on the other by the ergodic theorem of the reviewer. In fact the ergodic theorem allows one to conclude at once that the ordinary xth Fourier constant of $F(t)$ or "amplitude function" $a(x)=M_t[e^{-ixt}F(t)]$ exists for almost every P, as well as the "correlation function" $c(s)=M_t[F(t+s)\bar{F}(t)]$ if $q\geqq2$. The mean ergodic theorem of von Neumann and new Tauberian theorems also play an essential role in the paper.

Some of the most important results (stated here only for bounded measurable functions) are the following: If $F(t)$ of class (L^2) has an xth Fourier constant $a(x)$, for some x, and has a continuous correlation function $c(s)$ with associated "periodogram" $\varphi(x)$, defined uniquely as monotonic with $\varphi(x-0)=\varphi(x)$ by the equation

$$c(s)=\int_{-\infty}^{\infty} e^{-isx}d\varphi(x),$$

then $|a(x)|^2$ never exceeds the jump of $\varphi(x)$ at x, and equals it if and only if the "commutability condition"

$$[M_sM_t-M_tM_s](e^{-isx}F(t+s)\bar{F}(t))=0$$

is satisfied [theorem 1]. By an earlier result of the authors and its completion, the amplitude function $a(x)$ will certainly exist and vanish at points of continuity of $\varphi(x)$ [see theorem 2 and the addendum]. Necessary and sufficient conditions that $F(t)$ be almost periodic (B^2) are: $a(x)$ and a continuous $c(s)$ exist with $\varphi(x)$ a step function, and the commutability condition is satisfied for all x [theorem 3]. In the case $F(t)=f(T^tP)$, with periodogram $\varphi_{Pf}(x)$ for almost all P, if there exists an enumerable set of points which contains for almost all P the points of discontinuity of $\varphi_{Pf}(x)$, then $F(t)$ is almost periodic (B^2) if and only if $\varphi_{Pf}(x)$ is a step function for almost all P [theorem 4].

In the remainder of the paper it is further assumed that there exists a complete orthogonal system on S, in order to relate the paper to the results of Carleman, Koopman and Stone concerning unitary transformations $U^t(f)=f(T^tP)$ in the space of functions $f(t)$. A typical result is that the almost periodicity B^2 means precisely that the unitary flow has a pure point spectrum [theorem 6] containing for almost all P the Fourier exponents of $f(T^tP)$ [theorem 7]. The important case of metrically transitive flow is specially discussed; here, roughly speaking, the spectrum X is proved to be independent of P and f [theorems 7 and 8].

G. D. Birkhoff (Cambridge, Mass.).

Birkhoff, G. D. What is the ergodic theorem? Amer. Math. Monthly 49, 222–226 (1942). [MF 6509]

The present article is one of the series of articles invited by the American Mathematical Monthly, to describe in a simple and concise form some most striking developments in modern mathematics. The ergodic theorem, the proof of which was given in 1931 by the author of the present article, represents one of the most important achievements in mathematics in the twentieth century. In the present article the author, after a simple description of the theory of measure, gives a statement of the ergodic theorem in the case of measure preserving transformations of an interval into itself, and indicates its various generalizations. The end of the article is devoted to applications of the ergodic theorem to dynamical systems, in particular to the theory of a convex billiard table, on which a billiard ball moves with the velocity 1. *J. D. Tamarkin* (Providence, R. I.).

Theory of Probability

★Göring, Emil. Eine Erweiterung der Mises'schen Kollektive und der entsprechende Ausbau der Theorie der Wahrscheinlichkeitsrechnung. Transactions of the Twelfth International Congress of Actuaries, Lucerne, 1940. Vol. I, pp. 329–348. Orell Füssli, Zurich, 1941. (French, Italian, English summaries)

The author introduces the notion of a "generalized collective" as an extension of the von Mises collective. A generalized collective is a sequence $\{a_n\}$ $(n=1, 2, \cdots)$ of observations, where each element a_n can be considered as a single observation drawn from a von Mises collective K_n. If the collectives $K_1, K_2, \cdots$ are all equal to each other, the generalized collective reduces to a von Mises collective. The author thinks that his generalization is necessary for the treatment of certain problems in vital statistics. The

reviewer cannot share this opinion and believes that the generalization given by the author is merely an unnecessary complication of the theory. *A. Wald.*

Kemble, Edwin C. Is the frequency theory of probability adequate for all scientific purposes? Amer. J. Phys. 10, 6–16 (1942). [MF 6397]

Ferrari, Esther. On Bertrand's paradox. Union Mat. Argentina, Publ. no. 19, 15 pp. (1941). (Spanish) [MF 7141]

Identical with Revista Union. Mat. Argentina 7, 1–6, 74–80 (1940–1941); these Rev. 2, 227; 3, 168.

Geiringer, Hilda. A note on the probability of arbitrary events. Ann. Math. Statistics 13, 238–245 (1942). [MF 6926]

This note is concerned with the theory of arbitrarily linked events. The author indicates simple proofs for two theorems by Chung. She also gives a geometrical treatment of the problem of moments for a discrete distribution arising from a system of linked events. *A. H. Copeland.*

Leser, C. E. V. Inequalities for multivariate frequency distributions. Biometrika 32, 284–293 (1942). [MF 6528]

Let $y(x_1, \cdots, x_n)$ be the joint probability density function of the variates $x_1, \cdots, x_n$. The expected values of $x_1, \cdots, x_n$ are assumed to be zero and the standard deviation of x_i $(i=1, \cdots, n)$ is denoted by σ_i. Let P be the probability that the inequality

$$\left(\frac{x_1}{\lambda_1\sigma_1}\right)^2+\cdots+\left(\frac{x_n}{\lambda_n\sigma_n}\right)^2\leqq n$$

holds, where $\lambda_1, \cdots, \lambda_n$ are constants. In this paper several inequalities for P are derived under certain conditions on the density function $y(x_1, \cdots, x_n)$. If $1/\lambda_1^2+\cdots+1/\lambda_n^2\leqq n$, then the inequality $P\geqq 1-n^{-1}(1/\lambda_1^2+\cdots+1/\lambda_n^2)$, which is a generalization of Tchebycheff's inequality, is shown to hold without any restrictions on the density function $y(x_1, \cdots, x_n)$. *A. Wald* (New York, N. Y.).

Curtiss, J. H. On the distribution of the quotient of two chance variables. Ann. Math. Statistics 12, 409–421 (1941). [MF 6051]

The problem of determining the distribution of the quotient of two random variables has been solved in a considerable number of special cases, but the present paper is the first exhaustive investigation of the problem. The author considers in detail the question of general conditions for the existence of the distribution of the quotient of two random variables. He then proceeds to derive general expressions for (a) the probability function of the quotient of two random variables having an absolutely continuous joint probability function, (b) the distribution function of the ratio of a pair of arbitrary independent random variables in terms of the distribution functions of the two variables, (c) the distribution function of the ratio of two arbitrary independent random variables in terms of the characteristic functions of the two variables, (d) the limiting form of the distribution function of a quotient of two sums of arbitrary identically and independently distributed random variables. Expressions similar to those described in (a) and (b) are derived for the product of two random variables. Some of the expressions are illustrated by well-known special cases. *S. S. Wilks* (Princeton, N. J.).

Lukacs, Eugene. A characterization of the normal distribution. Ann. Math. Statistics 13, 91–93 (1942). [MF 6388]

The stochastic independence of the mean $\bar{x}$ and the variance s^2 of a sample of a random variable x is known to be necessary and sufficient for the normality of the probability distribution of x. A new proof, using characteristic functions, is given for this theorem, and a generalization to the multivariate case is indicated. *Z. W. Birnbaum.*

Rosenblatt, Alfred. Sur les théorèmes des grands nombres dans la théorie de la probabilité. Actas Acad. Ci. Lima 3, 152–159 (1940). [MF 7110]

The author discusses the closeness of the approximation of the usual success ratio to its limiting probability (in the Bernoulli case), comparing the results using Tchebycheff's formula and generalizations involving moments of the fourth order. *J. L. Doob* (Washington, D. C.).

Dieulefait, C. E. Some new derivations of limiting probability functions. Univ. Nac. Tucumán. Revista A. 2, 25–30 (1941). (Spanish) [MF 6746]

The author supposes that, among m elements, s have property E, and $r=m-s$ have property F. Following Pólya, he supposes that, when an element is drawn, it is replaced along with Δ elements of the same kind. With j a whole number, let $n\{j, \Delta\}$ be the number of drawings to be made to get exactly j of the E's. The probability $P(n\{j, \Delta\})$ for this can be expressed in terms of the binomial coefficient C_{n-j}^{n-1} and such products as $r_{n-j}=r(r+\Delta)\cdots(r+\{n-1-j\}\Delta)$, and also with the aid of gamma functions and a beta function. With the aid of the characteristic function $\sum_{n=j}^{\infty}P(n\{j, \Delta\})e^{nt}$, the expected values or moments are expressed. From $L(u)=B^{-1}(p', q')u^{q'-1}(1+u)^{-m/\Delta}$, there is obtained, as an approximation for large j, $E[(n-j)/j]^k =\int_0^{\infty}L(u)u^k du$, which is, indeed, exact for $k=1$. The case of $n(j, -\Delta)$ is also treated. Here, in particular, $n(j, -1)$ implies simple nonreplacement, but $n(j, 0)$ implies simple replacement. In the latter case, the author's method leads for large j to the well-known q/p as the value which $(n-j)/j$ will approach almost with certainty. *E. L. Dodd.*

Kozakiewicz, W. Sur la convergence presque certaine. Bull. Sci. Math. (2) 64, 121–128 (1940). [MF 6786]

Let $\{X_n\}$ be a sequence of random variables and let

$$F_{np}(x)=\text{Prob }\{X_{n+1}<x, \cdots, X_{n+p}<x\},$$
$$H_{np}(x)=\text{Prob }\{X_{n+1}\geqq x, \cdots, X_{n+p}\geqq x\}.$$

If X is a random variable having $F(x)$ as its distribution function then

$$\lim_{n\to\infty}\lim_{p\to\infty}F_{np}(x)=\lim_{n\to\infty}\lim_{p\to\infty}[1-H_{np}(x)]=F(x)$$

at each point of continuity of $F(x)$ is a necessary and sufficient condition for convergence almost everywhere of $\{X_n\}$ to X. Several related results concerning convergence almost everywhere are given. *M. Kac* (Ithaca, N. Y.).

Erdös, Paul. On the law of the iterated logarithm. Ann. of Math. (2) 43, 419–436 (1942). [MF 7000]

Let

$$\epsilon_1(t)/2+\epsilon_2(t)/2^2+\cdots$$

be the dyadic expansion of the real number t $(0\leqq t\leqq 1)$ and put

$$f_n(t)=\sum_{k=1}^{n}\epsilon_k(t)-n/2.$$

The author refines the well-known law of the "iterated logarithm" as follows: If

$$\varphi(n)=\left(\frac{n}{2\log\log n}\right)^{\frac{1}{2}}(\log\log n+\tfrac{3}{2}\log_3 n+\tfrac{1}{2}\log_4 n+\cdots+\tfrac{1}{2}\log_{k-1} n+(\tfrac{1}{2}+\delta)\log_k n),\quad k>3,$$

and $\delta>0$, then $f_n(t)>\varphi(n)$ almost everywhere can hold for a finite number of n's only. On the other hand if $\delta<0$ there is an infinite sequence of n's for which $f_n(t)>\varphi(n)$ almost everywhere. The methods employed are elementary, although by no means simple. *M. Kac* (Ithaca, N. Y.).

Opatowski, I. An inverse problem concerning a chain process. Proc. Nat. Acad. Sci. U.S.A. **28**, 83–88 (1942). [MF 6328]

Consider a Markoff chain with the possible states E_i, $i=0, 1, \cdots, n$, and depending on a continuous time-parameter t. It is supposed that only transitions $E_i\rightarrow E_{i+1}$ are possible, and that the conditional probability of such a change during any time interval dt is $p_i dt+o(dt)$, where $p_i>0$ is a constant. At $t=0$ the state E_0 existed. Then, as is well known, the probability $P_i(t)$ of the system being in state E_i at time t satisfies the differential equation $P'_i(t)=-p_iP_i(t)+p_{i-1}P_{i-1}(t)$, with $P_0(0)=1$, $P_i(0)=0$ for $i>1$ and $p_{-1}=0$. The author observes that in many applications $P_n(t)$ is the only one among the $P_i(t)$ which can be obtained from empirical observations. Hence the problem arises of evaluating the constants p_i and the number n knowing only $P_n(t)$. The author gives three methods for the solution of this problem. His main tools are the Laplace transform and the complete homogeneous symmetric functions (or homogeneous product sums). *W. Feller.*

Blackwell, David. Idempotent Markoff chains. Ann. of Math. (2) **43**, 560–567 (1942). [MF 7010]

Let $P_n(x, E)$ be the transition probability function of a Markoff chain, the probability of a transition (in n steps) into a point of the set E if the initial point is x. The author discusses idempotent transition probability functions $P_1(x, E)$: those supposed to satisfy the condition $P_1\equiv P_2\equiv P_3\equiv\cdots$. In the general case, the limiting functions of $(1/N)\sum_1^N P_n$ as $N\rightarrow\infty$ will define such idempotent functions. These idempotent functions have been described completely in the special case of a denumerable x-space by Yosida and Kakutani [Jap. J. Math. **16**, 47–55 (1939); these Rev. **1**, 62] and by the reviewer [see the following review]. The author shows that the results in the above special case can be extended to considerably more inclusive cases. The conditions he uses are satisfied, for instance, if the probability $P_1(x, E)$ can be expressed as the integral of a density function, or if x-space is Euclidean n-space, and if E runs through the Borel sets of this space. An example is given to show that such an extension is not possible in all cases.
J. L. Doob (Washington, D. C.).

Doob, J. L. Topics in the theory of Markoff chains. Trans. Amer. Math. Soc. **52**, 37–64 (1942). [MF 6993]

This first half of the paper is devoted to the study of matrices $P(t)=(p_{ij}(t))$ such that

$$(1)\qquad p_{ij}(t)\geq 0,\quad \sum_j p_{ij}(t)=1,\quad P(s)P(t)=P(s+t).$$

Such matrices are of obvious importance in the theory of Markoff chains. Doeblin [Bull. Sci. Math. (2) **62**, 21–32 (1938); **63**, 23–32, 35–64 (1939), in particular, pp. 35–37; also Thesis, Paris, 1938] discussed regularity and asymptotic properties of matrices $P(t)$ in the finite dimensional case. The author develops methods that are also applicable to infinite matrices of type (1). The statements of most theorems are too involved to be reproduced here. We mention only the following result which is both simple and striking: If $p_{ij}(t)$ satisfy (1) and are continuous,

$$\lim_{T\to\infty} T^{-1}\int_0^T p_{ij}(t)dt=\liminf_{t\to\infty} p_{ij}(t)$$

for every pair (i, j) of subscripts. The second half of the paper [pp. 49–64] deals with actual transitions connected with Markoff chains. *M. Kac* (Ithaca, N. Y.).

Doob, J. L. The Brownian movement and stochastic equations. Ann. of Math. (2) **43**, 351–369 (1942). [MF 6466]

On the background of the physical ideas of Einstein, Smoluchowski, Ornstein, Uhlenbeck and Langevin, and of the mathematical notions of P. Lévy and N. Wiener, the author studies one-dimensional Brownian movement on the usual basis of the introduction of $\mathbf{u}(t)$ and $\mathbf{x}(t)$, two 1-parameter variates corresponding to the observed velocity and position at epoch t of the particle. He achieves notable simplicity by proceeding in the following two stages:

First, the statistics of the velocity variate $\mathbf{u}(t)$ are introduced on the basis of the following theorem: Hypothesis (1). The statistics of $\mathbf{u}(t)$ is unchanged by t-translation. Hypothesis (2). $\mathbf{u}(t)$ is Markoffian, that is, for any $t_1<\cdots<t_n$, the relative distribution of $\mathbf{u}(t_n)$ for given values $\mathbf{u}(t_1), \cdots, \mathbf{u}(t_{n-1})$ depends only on $\mathbf{u}(t_{n-1})$. Hypothesis (3). For any distinct s, t, variates $\mathbf{u}(s)$, $\mathbf{u}(t)$ form a bivariate normal distribution. Conclusion. For any $t_1<\cdots<t_n$, the variates $\mathbf{u}(t_1), \cdots, \mathbf{u}(t_n)$ are either (A) mutually independent and normal; or else (B) n-variate normal, the correlation coefficients of $\mathbf{u}(s)$, $\mathbf{u}(t)$ being given by $e^{-\beta|t-s|}$ (β independent of s, t). Case (A) leads to the Einstein-Smoluchowski distribution; (B) is the Ornstein-Uhlenbeck case, which is taken as fundamental in this paper.

Secondly, the distribution for the position variate is introduced in a manner suggested by the formula $\mathbf{x}(t)-\mathbf{x}(0)=\int_0^t\mathbf{u}(s)ds$. Now since this involves the integration of variates, it is not self-explanatory. For its interpretation the whole basis of the discussion is deepened by picturing $\mathbf{u}(t)$, not as infinitely many variates each taking as value a point in one-dimensional space, but as a single variate taking as "value" $\mathbf{u}(t)$ a point in function-space. This, the author emphasizes, corresponds perfectly with the physical picture: the particle determines a path at random (for example, in velocity-space), not merely a position at epoch t. A probability measure is introduced into this function-space by mapping it on Wiener's differential space and taking over the measure in the latter. Moreover, many of Wiener's theorems (of the "true almost everywhere" type) are simultaneously secured by this mapping, chief of which is that, with probability 1, the value $u(t)$ of $\mathbf{u}(t)$ is continuous for all t; hence $\int_0^t\mathbf{u}(s)ds$ exists. This is the sense of the above formula introducing $\mathbf{x}(t)$.

The remainder of the paper concerns the Langevin equation $du/dt=-\beta u(t)+A(t)$, which (in accordance with the analytical necessities) is paraphrased to the form $d\mathbf{u}(t)=-\beta\mathbf{u}(t)dt+d\mathbf{B}(t)$. The new variate $\mathbf{B}(t)$ is interpreted by calling $\mathbf{B}(t)-\mathbf{B}(s)$ the total random impulse (by surrounding molecules) between epochs s and t (reduced to suitable standard velocity). More precisely, the above

equation means that for all a, b, continuous $f(t)$, and almost all $u(t)$ (value of $\mathbf{u}(t)$),

$$\int_a^b f(t)du(t) = -\beta\int_a^b f(t)u(t)dt + \int_a^b f(t)dB(t),$$

the Riemann-Stieltjes integrals existing in the sense of Wiener. Upon these bases, the statistics of $\mathbf{u}(t)$, especially property (B) above, are related to the statistics of $\mathbf{B}(t)$, in particular, those properties resulting from the general nature of the Maxwell distribution and its generalizations.
B. O. Koopman (New York, N. Y.).

Hartman, Philip and Wintner, Aurel. On the infinitesimal generators of integral convolutions. Amer. J. Math. **64**, 273–298 (1942). [MF 6431]

The authors discuss in detail two classes of solutions of the functional equation (I) $\sigma_s(x)*\sigma_t(x)=\sigma_{s+t}(x)$, in which $\sigma_s(x)$ for each $s>0$ is a distribution function in x, and the left side of (I) is the convolution of the distribution functions concerned. The general solution of (I) was obtained by Kolmogoroff [Atti Accad. Naz. Lincei. Rend. (6) **15**, 805–808, 866–869 (1932)] under the restriction of finite second moments, and without this restriction by Lévy [Ann. Scuola Norm. Super. Pisa (2) **3**, 337–366 (1934)]. Solutions of (I) determine an important class of stochastic processes. Lévy's general solution is (apart from a centering term) the convolution of a pair of the solutions discussed by Hartman and Wintner. Only one of their two classes will be noted here. If $\mu^0(x)$ is a monotone function, increasing from 0 to a, continuous on the right, whose nth iterated convolution is μ^n, the series $e^{-at}\sum_0^\infty t^n\mu^n/n!$ defines a distribution function $\sigma_t(x)$ satisfying (I). The authors investigate the details of the relation between monotone functions $\mu(x)$, $\sigma_1(x)$; one has a singular component if and only if the other has, etc. *J. L. Doob* (Princeton, N. J.).

Theoretical Statistics

***Wolfenden, Hugh H. The Fundamental Principles of Mathematical Statistics.** Macmillan Company of Canada Limited, Toronto, Ont., 1942. xv+379 pp. $5.00.

This book has been written with a view to satisfying the needs of actuaries and vital statisticians. Thus emphasis is given to those portions of the theory and applications which are most frequently used in actuarial practice. The book contains the following chapters: (I) Introduction; (II) The nature of the problems; (III) The classical approach; (IV) The combination of observations; (V) The theory of random sampling; (VI) Generalization of the binomial law—the "multinomial" distribution; (VII) Frequency distributions and curves in general; (VIII) The fitting of curves, and graduation; (IX) The tests of goodness of fit; (X) Recent researches, and miscellaneous problems; (XI) An outline of a course in graduation. In addition to the above chapters, three sections are included. Section A is devoted to the history of statistics. In section B the author elaborates on the mathematical analyses underlying some of the statistical procedures. In section C the theory is illustrated by a number of applications.

The book contains very little on the modern developments in the theory of estimation and testing hypotheses. On the credit side one may mention, in particular, the chapters on frequency distributions, fitting of curves and graduation which meet especially the requirements of the actuary. The reader will also welcome the numerous references given in the text and the bibliography included at the end of the book. *A. Wald* (New York, N. Y.).

***Holzinger, Karl J. and Harman, Harry H. Factor Analysis. A synthesis of factorial methods.** University of Chicago Press, Chicago, Ill., 1941. xii+417 pp. $5.00.

This book is devoted primarily to a clear exposition of the theory of factor analysis in the state to which it has now been brought and of the manner of its numerical application. It gives relatively little space to the interpretation of the results in any particular field of psychology which is admittedly a matter for caution. The student with more than a minimum of mathematical preparation and a knowledge of mathematical statistics will find this work a clear and felicitous road to a good grasp of its subject, but I am more uncertain of the ease with which a mathematically uninitiated worker in psychology, say, will use it as a laboratory manual. To this second class of readers I believe that the directions will appear somewhat economical of statement and that the discussion will seem somewhat mathematically sophisticated.

There are four principal sections. In the first the underlying mathematical framework and the basic statistical tools are set forth. In particular, the useful and enlightening geometrical interpretation of the whole theory is especially well explained and is well used throughout the remainder of the book. It is emphasized that there is no unique resolution into factors and a list of standards is set down for assistance in choosing a mode of factorization. In part II orthogonal solutions obtained directly, uni-factor, bi-factor, principal-factor and centroid, are derived and the numerical steps in carrying them out are illustrated. The preliminary estimation of communalities is important here and several methods for accomplishing this are given. A variant of the centroid method is shown which may be tried if good estimates of the communalities are not available. Some of the desirable sampling formulae are developed and discussed though as it is stated the existing theory is far from being satisfactorily complete. Perhaps a somewhat fuller explanation of the shortcomings of the formulae given would have been desirable. In part III the use of orthogonal linear transformations is developed, by which derived principal-factor and multiple-factor solutions may be obtained from more simply calculated solutions. Here, too, it is shown how one may arrive at an oblique solution with a possible reduction in the number of group factors from an orthogonal solution. In part IV methods for the actual estimation of factors are given and the relations between different forms of solutions are discussed. Finally the appendix contains some further mathematical proofs omitted from the text and includes detailed instructions for the numerical steps also not given in the text.

The scientific caution and restraint shown by the authors throughout is commendable. The claims of factor analysis as a tool for scientific investigation are fairly stated and not overstated. The book as a whole is an excellent and valuable job. *C. C. Craig* (Ann Arbor, Mich.).

***Molina, E. C. Poisson's Exponential Binomial Limit. Table I: Individual Terms. Table II: Cumulated Terms.** Van Nostrand Company, Inc., New York, 1942. v+47 pp. $2.75.

The lack of really extensive tables of Poisson's distribution function has been felt badly by many statisticians and

engineers. The present tables will be particularly useful since their range is extremely wide. They have been prepared, and used, in the Bell Telephone Laboratories with a view to application to problems of telephone trunking as well as to quality inspection problems. It is unnecessary to point out that the use of these tables is much wider. Table I gives, for every n, values of $P_n(a)=a^n e^{-a}/n!$ to six decimal places for $0\leqq a\leqq 16$ in steps of .1 and for all integer values $16\leqq a\leqq 100$; moreover, seven decimal places are given for $0\leqq a\leqq .01$ in steps of .001 and for $.01\leqq a\leqq .30$ in steps of .01. Table II gives the corresponding sums $P(N,a)=\sum_{n=N}^{\infty} a^n e^{-a}/n!$. [The tables were first reproduced in 1938 for use in the Bell System.] *W. Feller.*

***Tables of Probability Functions. Vol. II.** Prepared by the Federal Works Agency, Work Projects Administration for the City of New York, as a Report of Official Project No. 165-2-97-22; conducted under the sponsorship of the National Bureau of Standards. Technical Director: Arnold N. Lowan. New York, 1942. xxi+344 pp. $2.00.

The first volume of these useful tables was reviewed in these Rev. **3**, 5. The present volume is similar in arrangement and range, but is concerned with the normalized probability function

$$\Phi(x)=(2\pi)^{-\frac{1}{2}}\int_{-x}^{x} e^{-\alpha^2/2}d\alpha.$$

Values of $\Phi(x)$ and its derivative are given to 15 decimal places for $0\leqq x\leqq 1$ in steps of .0001, for larger x in steps of .001. A second table gives $1-\Phi(x)$ and $\Phi'(x)$ to seven significant figures for $6\leqq x\leqq 10$ in steps of .01. A short introduction describes the use of the tables. *W. Feller.*

Birnbaum, Z. W. An inequality for Mill's ratio. Ann. Math. Statistics **13**, 245–246 (1942). [MF 6927]

The inequality recently proved by R. D. Gordon [Ann. Math. Statistics **12**, 339–343 (1941); these Rev. **3**, 171] is replaced by

$$\tfrac{1}{2}[(4+x^2)^{\frac{1}{2}}-x]e^{-x^2/2}<\int_x^{\infty} e^{-t^2/2}dt.$$

The proof uses properties of convex functions. [It may be noted that sharper and simpler asymptotic estimates are known, which can be proved by elementary methods. See, for example, Lowan's introduction to Tables of Probability Functions, vol. I, New York, 1941; these Rev. **3**, 5.]
W. Feller (Providence, R. I.).

Pearson, E. S. The probability integral of the range in samples of n observations from a normal population. I. Foreword and tables. Biometrika **32**, 301–308 (1942). [MF 6531]

Hartley, H. O. The probability integral of the range in samples of n observations from a normal population. II. Numerical evaluation of the probability integral. Biometrika **32**, 309–310 (1942). [MF 6532]

Moyal, J. E. Approximate probability distribution functions for the sum of two independent variates. J. Roy. Statist. Soc. **105**, 42–43 (1942). [MF 7066]

Using Taylor's formula it is seen that the frequency distribution of a sum X_1 and X_2 is formally represented by $\sum(-1)^k m_k f^{(k)}(x)/k!$, where $f(x)$ is the frequency distribution of X_1 and the m_k denote the moments of X_2. Instead of derivatives successive differences may be used. The author applies this result to the fitting of empirical frequency distributions. *W. Feller* (Providence, R. I.).

Burr, Irving W. Cumulative frequency functions. Ann. Math. Statistics **13**, 215–232 (1942). [MF 6923]

To express probability P the author uses the cumulative frequency function $F(x)$ and for the continuous case writes $P(a\leqq x\leqq b)=F(b)-F(a)$; for the discrete case, $P(a\leqq x\leqq b)=F(b+h)-F(a)$, where h is the interval between consecutive abscissas. However, in the continuous case, instead of taking the jth moment about a as $\int_{-\infty}^{\infty}(x-a)^j dF(x)$, he takes it as

$$\int_a^{\infty}(x-a)^j[1-F(x)]dx-\int_{-\infty}^{a}(x-a)^j F(x)dx,$$

having set up contact conditions for the existence of these integrals. For the discrete case he gives a similar difference, using, however, in place of $(x-a)^j$, the incomplete factorial $(x-a)(x-a-h)\cdots(x-a-\overline{j-1}h)$. These new moments are then expressed in terms of the usual moments. In his scheme of curve-fitting the parameters for location and scale are determined last, the other or "primary" parameters first. With $y=F(x)$, and $dy/dx=y(1-y)g(x,y)$, several special cases of $g(x,y)$ are studied. The function $F(x)=1-(1+x^c)^{-k}$ is treated in some detail, with numerical illustrations. *E. L. Dodd* (Austin, Tex.).

Kärsna, Aarne. Über das System der einmodigen Häufigkeitskurven. Acta Comment. Univ. Tartuensis. A. **35**, no. 1, 65 pp. (1940). [MF 6375]

In criticism of the Pearson types the author states that the parameters of these types do not reveal easily the chief characteristics of the distribution such as range, skewness, etc., and that the computations are very lengthy and difficult. Moreover, the author gives an example of a distribution in which, as the mean ordinate is changed from 40 to 41, the Pearson criterion calls for five different Pearson types, stating that such a change would be purely accidental unless the total frequency exceeds 5560.

The author proposes a system of five-parameter frequency functions given by

$$y=A\{1+\cos \pi[\log (a+bx)/\log \beta]^n\}. \tag{72}$$

Here A, a and b are parameters for scale and location, β for skewness, n for sharpness. The curves have but one mode, and extend from $x=(1-a\beta)/b\beta$ to $x=(\beta-a)/b$. That portion of the range which remains after 5% of the total frequency is excluded from each end is called the basis. Three indicators are then introduced, $P(\%)$, $M(\%)$ and μ, as follows. The basis AB is trisected. If the mode is nearer A, $P(\%)$ is the proportion of the frequency which lies above the left third of the basis, otherwise, above the right third. Let M be the median. If the mode is nearer A, then $M(\%)=AM/AB$, otherwise, MB/AB. Let H^1 be the height of a horizontal line which cuts off an upper 5% of area beneath the frequency curve. Let h be the total frequency divided by the basis. Then $\mu=H^1/h$. Tables are given for β and n and $P(\%)$; for β, n and $M(\%)$; and for β, n and μ. Nomograms are given, including one for β and n from $P(\%)$ and μ, and one for β and n from $M(\%)$ and μ.

For distributions of the temperature at Tartu 1866–1935, the curves of the author appear to give a better fit than those of Pearson. *E. L. Dodd* (Austin, Tex.).

Raiford, Theodore E. Skewness of combined distributions. J. Amer. Statist. Assoc. **37**, 391–393 (1942). [MF 7145]

As a measure of skewness the author takes the quotient of the third unit moment by the cube σ^3 of the standard deviation. He refers to a former paper [Human Biology, Feb., 1938, p. 139], in which appears the main formula developed in the present paper to express the measure of skewness of a set in terms of measures $\alpha_{3:i}$ of skewness of subsets together with the standard deviations σ_i and the differences $d_i = M_i - M$ between the means of subsets and the mean M of the whole set. With $\sum n_i = N$,

$$\alpha_3 = (N\sigma^3)^{-1}[\sum n_i\sigma_i^3\alpha_{3:i} + \sum n_i d_i(3\sigma_i^2 + d_i^2)].$$

The author points out that the measures of skewness of the subsets may be all of one sign while that of the set is of opposite sign. *E. L. Dodd* (Austin, Tex.).

Simaika, J. B. Interpolation for fresh probability levels between the standard table levels of a function. Biometrika **32**, 263–276 (1942). [MF 6526]

Let x be a random variable and for any positive $\alpha \leqq 1$ let x_α be the value of x for which the probability that $x < x_\alpha$ is equal to α. The author deals with the problem of interpolating x_α if x_α is known for a few neighboring values of α. For this purpose the variate

$$u = \frac{x - \text{mean of } x}{\text{standard deviation of } x}$$

is introduced. Denote by U a normally distributed variate with zero mean and unit variance. On the basis of a series expansion of the distribution of u the author finds that u_α can be approximated by a fifth degree polynomial of U_α if the rth cumulant of u is negligible for $r \geqq 5$. The coefficients of this polynomial are functions of the third and fourth cumulants of u. For practical applications it is proposed to interpolate u_α as a function of U_α using the Lagrange polynomial interpolation formula. This interpolation method is applied to the χ^2, t and beta-distributions and the interpolated values are numerically compared with the exact values. The author finds that for many purposes linear interpolation is adequate; for others a second order Lagrange formula may be preferred. However, no general mathematical theory is given as to the accuracy of the approximation methods proposed. *A. Wald.*

Lidstone, G. J. Notes on the Poisson frequency distribution. J. Inst. Actuar. **71**, 284–291 (1942). [MF 6523]

The author considers the Poisson distribution $\psi(n) = e^{-a}a^n/n!$ as an approximation to the binomial. As he points out, it is known [cf., for example, A. C. Aitken, Statistical Mathematics, Oliver and Boyd, Edinburgh, 1939; these Rev. 1, 247] that the corresponding Charlier type B series $\psi(n) + B_2\Delta^2\psi(n-2) + B_3\Delta^3\psi(n-3) + \cdots$ will give still better approximations. The author shows by numerical examples that in many cases this series gives a better fit than Pearson's type III curves. Moreover, he studies in detail the behavior of $\psi(n)$ and its successive differences near the mode. *W. Feller* (Providence, R. I.).

Finney, D. J. On the distribution of a variate whose logarithm is normally distributed. Suppl. J. Roy. Statist. Soc. **7**, 155–161 (1941). [MF 6539]

The author writes: "For a number of biological, and other, populations the standard error of an individual observation appears to be approximately proportional to the magnitude of the observation. In such cases the data prove more amenable to statistical treatment if first transformed by taking the logarithm of each observation." Starting from $x = \log y$, where x is normally distributed with mean ξ and variance σ^2, the expected value $E(y^r)$ of y^r is found to be $\exp(r\xi + r^2\sigma^2/2)$, where $\exp(t) = e^t$. And thus, if $\tau = \exp(\sigma^2)$, then $\mu_2(y) = \mu^2(\tau - 1)$. And, if $x = S(x)/n$, $s^2 = S(x-\bar{x})^2/(n-1)$, and p is integral, then

$$E(s^{2p}) = (n+1)(n+3)\cdots(n+2p-3)(n-1)^{-p-1}\sigma^{2p}.$$

Setting

$$g(t) = 1 + (n-1)(t/n) + \sum_2^\infty (n-1)^{2r-1}(t/n)^r[r!(n+1)(n+3)\cdots(n+2r-3)]^{-1},$$

it is found that $E\{e^{rx}g(r^2s^2/2)\} = E(y^r)$; and thus that efficient estimates of the mean and variance of the y population are $m = e^{\bar{x}}g(s^2/2)$, and $v = e^{2x}\{g(2s^2) - g[(n-2)s^2/(n-1)]\}$. The "Summary" states: "Formulae have been obtained for efficient estimates of the mean and variance of a population from a sample, when it is known that the logarithm of an observation is normally distributed. As the exact expressions are not very suitable for arithmetical computation, approximations are developed which may be used for moderately large samples. . . ." *E. L. Dodd* (Austin, Tex.).

Haldane, J. B. S. Moments of the distributions of powers and products of normal variates. Biometrika **32**, 226–242 (1942). [MF 6524]

It is known that frequently distributions of weights are skewed while the distributions of the corresponding linear measurements appear normal. To clarify the circumstances under which this happens, the author investigates the probability distributions of powers of normal variables, as well as of products of independent and of correlated normal variables. In each case the first few moments and cumulants of the distribution are computed, and the skewness and the excess are expressed in terms of the coefficients of variation of the original variables. The following are some of the results obtained: the distribution of the cube of a normal variable has positive skewness and positive excess; the distribution of the product of three independent normal variables with small coefficients of variation is almost undistinguishable from the distribution of the cube of a normal variable; the distribution of the product of three correlated normal variables fulfilling certain conditions is almost independent of the coefficients of correlation, if the coefficients of variation of all the original variables are of the same order of magnitude. *Z. W. Birnbaum.*

Haldane, J. B. S. The mode and median of a nearly normal distribution with given cumulants. Biometrika **32**, 294–299 (1942). [MF 6529]

The mode and the median of a distribution are formally expressed as infinite series in terms of the moments or cumulants of the distribution. The convergence of the series is not discussed but it is pointed out that for some classes of distributions the series yield useful asymptotic expressions. Those expressions are used to investigate the conditions under which the approximate relationship

$$3\cdot(\text{median} - \text{mode}) = 2\cdot(\text{mean} - \text{mode}),$$

found empirically by Pearson for curves of type III, holds for various distributions. *Z. W. Birnbaum.*

Rajalakshman, D. V. On the extreme values of samples taken from a rectangular population. Math. Student 9, 103–111 (1941). [MF 6474]

For the mth moments μ_m and ν_m of the maximum and minimum values of samples of size n taken from an infinite rectangular population of range ω, about their respective arithmetic means, the author derives the following:

$$\mu_m=[\omega/(n+1)]^m\cdot F(-m, 1; n+1; n+1); \quad \nu_m=(-1)^m\mu_m;$$

$$F(a, b; c; z)=1+\frac{a\cdot b}{1\cdot c}z+\frac{a\cdot(a+1)\cdot b\cdot(b+1)}{1\cdot 2\cdot c\cdot(c+1)}z^2+\cdots,$$

this F being the hypergeometric series. For a first derivation, he uses a result of Karl Pearson [Biometrika 23, 364–397 (1931), in particular, p. 373]; and this is followed by a second derivation using a method similar to one adopted by N. S. Sastry. Experimental verification was obtained; and the frequency distributions of extreme values were of Pearson type IX. *E. L. Dodd* (Austin, Tex.).

Hartley, H. O. The range in random samples. Biometrika 32, 334–348 (1942). [MF 6535]

If the observations x_α ($\alpha=1, \cdots, n$) in a random sample of size n are arranged in ascending order of magnitude the range of the sample is defined by the expression $w_n=x_n-x_1$. If the observations are grouped in intervals of length h the range of the sample is given by $\xi_n-\xi_1$, where ξ_n denotes the center of the interval which contains the largest observation and ξ_1 denotes the center of the interval which contains the smallest observation. The author finds that the expected value of $\xi_n-\xi_1$ is given by

$$E(\xi_n-\xi_1)=h\lim_{j\to\infty}\left\{(2j+1)-\sum_{i=-j+1}^{j+1}\left(\int_{-\infty}^{i}\right)^n-\sum_{i=-j+1}^{j+1}\left(\int_{i}^{\infty}\right)^n\right\},$$

where $\int_a^b$ stands for $\int_{\xi+ah}^{\xi+bh}f(x)dx$, $f(x)$ denotes the probability density function of x and ξ denotes the distance of the expected value of x from the nearest group interval endpoint. The parameter ξ involved in the expected value of $\xi_n-\xi_1$ is usually unknown. In order to overcome this difficulty it is assumed that ξ is a random variable with a rectangular distribution in the interval $[-\frac{1}{2}h, \frac{1}{2}h]$. In this case it is shown that

$$P(n, h, m)=h^{-1}\int_{mh}^{(m+1)h}P_n(W)dW,$$

where $P(n, h, m)$ denotes the probability that $\xi_n-\xi_1\leqq mh$ and $P_n(W)$ denotes the probability that the range of the ungrouped sample is less than or equal to W. It is furthermore shown that the expected value of the $\xi_n-\xi_1$ is the same as the expected value of the range of the ungrouped sample, and that the variance of $\xi_n-\xi$ is greater than the variance of the range of the ungrouped sample by approximately $\frac{1}{6}h^2$. *A. Wald* (New York, N. Y.).

Geary, R. C. Inherent relations between random variables. Proc. Roy. Irish Acad. Sect. A. 47, 63–76 (1942). [MF 6610]

It is supposed that a set of k statistical variables are such that their exact measures x_i, $i=1, 2, \cdots, k$, precisely satisfy a linear relation $\sum_{i=1}^k a_i x_i=d$, but that the observed values X_i are subject to error so that $X_i=x_i+\xi_i$, $i=1, 2, \cdots, k$, in which the ξ_i's are independent of the x_i's and of each other. Then, given an indefinitely large number of sets of observed values $X_1, \cdots, X_k$, how may one determine the coefficients a_i? It is first shown that product semi-invariants of the X_i's are identical with those of the x_i's, and then that there exists an infinite set of linear relations with coefficients a_i among product semi-invariants of the same order for the x_i's. In case the x_i's obey a normal distribution law, the fact that all semi-invariants of order three or more vanish identically makes the problem stated indeterminate, but for anormal distributions in general the a_i's can be found. It is pointed out that classical linear regression theory is unsuited to the problem. The conditions on the ξ_i's may be relaxed to permit them to be dependent on each other, and it is shown that cases of nonlinear relationships can also be handled. The questions of practical applicability and the requisite sampling theory are left for further investigation. *C. C. Craig* (Ann Arbor, Mich.).

Girshick, M. A. Note on the distribution of roots of a polynomial with random complex coefficients. Ann. Math. Statistics 13, 235–238 (1942). [MF 6925]

The author proves a "rather well-known theorem on complex Jacobians" to the effect that the square of the modulus of such a determinant of the nth order is equal to a Jacobian of the $(2n)$th order with real partial derivatives. His equations (4) and (8) are to be taken as symbolic. He then considers the roots z_j of the equation $z^n-a_1z^{n-1}+\cdots+(-1)^na_n=0$, where the a_p are complex chance numbers. Here $a_1=\sum z_j$, $a_2=\sum z_jz_k$, etc. With $\bar{a}_p$ as the conjugate of a_p, the problem is then to transform a normal distribution involving $\sum a_p\bar{a}_p$ into a distribution in z_p and $\bar{z}_p$. For this purpose, the Jacobian of the a's with respect to the z's is expressed as $\sum\sum(z_p-z_q)$, $1\leqq p<q\leqq n$. However, it would seem that a product was intended here, instead of a sum, as printed. *E. L. Dodd* (Austin, Tex.).

von Neumann, John. Distribution of the ratio of the mean square successive difference to the variance. Ann. Math. Statistics 12, 367–395 (1941). [MF 6049]

Suppose $x_1, x_2, \cdots, x_n$ are n independent observations on a random variable x distributed normally with mean ξ and variance σ^2. Let $s^2=n^{-1}\sum_1^n(x_i-\bar{x})^2$, $\delta^2=(n-1)^{-1}\times\sum_1^{n-1}(x_{i+1}-x_i)^2$, where $\bar{x}$ is the mean of the n x's. Let $\eta=\delta^2/s^2$. Using a special generating function Williams [Ann. Math. Statistics 12, 239–241 (1941); these Rev. 3, 7] determined the first four moments of η. In the present paper von Neumann derives the exact sampling distribution of η. He first shows that the problem of determining the sampling distribution of η is equivalent to that of determining the distribution function of the expression $n(n-1)^{-1}\sum_1^{n-1}A_iy_i^2$, where the point $(y_1, y_2, \cdots, y_{n-1})$ is uniformly distributed over the spherical surface $\sum_1^{n-1}y_i^2=1$, where $A_i=4\sin^2(i\pi/2n)$ $(i=1, 2, \cdots, n-1)$, which, together with 0, are the characteristic values of the quadratic form

$$(n-1)\delta^2=\sum_1^{n-1}(x_{i+1}-x_i)^2.$$

The distribution of η is then deduced as a special case of the distribution of $\sum_1^{m-1}B_ix_i^2=\gamma$, say $\omega(\gamma)$, where the point $(x_1, x_2, \cdots, x_m)$ is uniformly distributed over the spherical surface $\sum_1^m x_i^2=1$, and $B_1\geqq B_2\geqq\cdots\geqq B_m (B_1>B_m)$. Moreover $\omega(\gamma)$ is observed to be the solution of

$$\int_{B_m}^{B_1}(\gamma-z)^{-m/2}\omega(\gamma)d\gamma=\prod_1^m(B_i-z)^{-\frac{1}{2}}.$$

Solutions of this equation are given under various assumptions regarding the B_i and m, some of which cover the case of the distribution of η itself. Finally, it is shown that s and η are independent in the probability sense. *S. S. Wilks.*

von Neumann, John. A further remark concerning the distribution of the ratio of the mean square successive difference to the variance. Ann. Math. Statistics 13, 86–88 (1942). [MF 6386]

In the paper reviewed above von Neumann obtained the distribution of $\gamma=\sum a_i x_i^2$, where the x's are uniformly distributed over the sphere $\sum x_i^2=1$. The derivation made in that paper, however, depended on the assumption that m is an even integer. In the present note the author extends his results to cover the case in which m is an odd integer. He then shows how the result applies to the problem of determining the distribution of the following ratio for even values of n

$$\sum_1^{n-1}(x_i-\bar{x})(x_{i+1}-\bar{x})\Big/\sum_1^{r}(x_i-\bar{x})^2,$$

where $x_1, \cdots, x_n$ are elements in a sample from a normal population and $\bar{x}$ is the sample mean. *S. S. Wilks.*

Hart, B. I. Tabulation of the probabilities for the ratio of the mean square successive difference to the variance. Note by John von Neumann. Ann. Math. Statistics 13, 207–214 (1942). [MF 6922]

Let $x_1, x_2, \cdots, x_n$ be the elements in a sample from a normal distribution. von Neumann [see the preceding reviews] has obtained the sampling distribution $\omega(r)dr$ of the ratio $r=\delta^2/s^2$, where $\delta^2=(n-1)^{-1}\sum_{i=1}^{n-1}(x_{i+1}-x_i)^2$ and $s^2=n^{-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$, where $\bar{x}$ is the average of the x's in the sample. In the present paper the integral $\int_0^k\omega(r)dr$ is evaluated for $n=4$, 5, 6, 7, 8, 9, 10, 11, 12, 15, 20, 25, 30, 40, 50 and 60 and for values of k by steps of .05 from 0 to values for which the integral has values between .06 and .10. There seems to be no systematic value of k for which the tabulation terminates, but the tabulation is sufficient to determine 1 percent and 5 percent significance points for r and for the values of n given above. *S. S. Wilks.*

Anderson, R. L. Distribution of the serial correlation coefficient. Ann. Math. Statistics 13, 1–13 (1942). [MF 6379]

The serial correlation for lag L and N observations $X_1, \cdots, X_N$ is defined by the expression

$${}_LR_N=\frac{X_1X_{L+1}+\cdots+X_{N-L}X_N+X_NX_L-(\sum X_i)^2/N}{\sum X_i^2-(\sum X_i)^2/N}.$$

In this paper the distribution of ${}_LR_N$ is studied under the assumption that $X_1, \cdots, X_N$ are normally and independently distributed about the same mean with unit variance. The author derives the exact and large sample distributions of ${}_1R_N$ and gives a table of the exact significance points for all values of $N\leqq 70$. For $N>70$ the large sample approximation can be used. It is proved that the distribution of ${}_LR_N$ is the same as that of ${}_1R_N$ if L and N are prime to each other. It is furthermore stated that the distribution of ${}_LR_N$ can be obtained for any L and N if it is known for values of L and N for which L is a factor of N. The author derives the distribution of ${}_LR_N$ and gives the significance points in the case when $N/L=2$, 3 and 4. The results obtained are useful in the statistical analysis of time series. *A. Wald.*

Kendall, M. G. Partial rank correlation. Biometrika 32, 277–283 (1942). [MF 6527]

Suppose a set of n objects has been subjected to two ranking schemes, say 1 and 2. We then have the following situation: ranking 1: $a_1, a_2, \cdots, a_n$; ranking 2: $b_1, b_2, \cdots, b_n$, where the set of a's and the set of b's each form a permutation of the integers 1, 2, $\cdots$, n. Now suppose we consider the order in which each pair of a's occur in ranking 1 and similarly for the corresponding b's in ranking 2. If the integers in each pair of a's (or b's) occur in increasing order, we associate a $+$ with that pair and if in decreasing order a $-$. Let S_1 be the number of pairs of objects in which the rankings have the same sign and S_2 the number having opposite signs. Kendall defines the rank correlation between rankings 1 and 2 as $\tau_{12}=(S_1-S_2)/N$, where $N=\binom{n}{2}$ the total number of pairs of integers in each of the two rankings. Without loss of generality the objects can be arranged so that $a_1=1$, $a_2=2$, $\cdots$, $a_n=n$. Each pair of a's in this case will have the same sign $(+)$. The value of τ_{12} is clearly invariant under permutations of the objects. Now suppose we have a third ranking (ranking 3) of the n objects. Assuming that the objects are arranged such that $a_1=1, \cdots, a_n=n$, let a, b, c, d be the number of pairs of objects in which rankings 2 and 3 have the following combinations of signs, respectively: $++$, $-+$, $+-$, $--$. Kendall defines the partial rank correlation between rankings 2 and 3 on ranking 1 as

$$\tau_{23\cdot1}=(ad-bc)/(+((a+b)(c+d)(a+c)(b+d))^{\frac{1}{2}})=(\chi^2/N)^{\frac{1}{2}},$$

where χ^2 is the ordinary mean square contingency for a four-fold table with entries a, b, c, d. Kendall shows that $\tau_{23\cdot1}$ and his ordinary rank correlation coefficients τ_{12}, τ_{13}, τ_{23} satisfy a formula analogous to that of ordinary correlation and partial correlation coefficients, namely

$$\tau_{23\cdot1}=\frac{\tau_{23}-\tau_{12}\tau_{13}}{(1-\tau_{12}^2)^{\frac{1}{2}}(1-\tau_{13}^2)^{\frac{1}{2}}}.$$

S. S. Wilks (Princeton, N. J.).

Koopmans, Tjalling. Serial correlation and quadratic forms in normal variables. Ann. Math. Statistics 13, 14–33 (1942). [MF 6380]

In this paper the stochastical process $x_t=\rho x_{t-1}+z_t$ $(t=1, 2, 3, \cdots)$ is considered, where z_t are independent drawings from a normal distribution with mean zero and standard deviation σ, and the coefficient ρ is an unknown constant of which the absolute value is less than 1. For estimating the coefficient ρ on the basis of the observed values $x_1, \cdots, x_T$, the use of the maximum likelihood estimate $\hat{\rho}$ of ρ is proposed which is shown to be a function of the three quadratic forms $l=x_1^2+x_T^2$, $m=x_1x_2+\cdots+x_{T-1}x_T$ and $n=x_2^2+\cdots+x_{T-1}^2$. For the simple problem of testing the hypothesis that $\rho=0$ it is sufficient to know the distribution of $m/(l+n)$. The author restricts himself to the latter case and studies the distribution of the ratio $r=q/p$, where $p=x_1^2+\cdots+x_T^2$ and q is a quadratic form in the variables $x_1, \cdots, x_T$ with characteristic values $K_1, \cdots, K_T$. The variates $x_1, \cdots, x_T$ are assumed to be independently and normally distributed with zero means and unit variances. The author derives an expression for the probability density $h(r)$ of r, the numerical calculation of which is very cumbersome except for small values of T. In the statistic q/p, used for testing the hypothesis that $\rho=0$, the quadratic form q is equal to $m=x_1x_2+\cdots+x_{T-1}x_T$. The author considers also the slightly different case when $q=\bar{m}=x_1x_2+\cdots+x_{T-1}x_T+x_Tx_1$, which was suggested by H. Hotelling. It is shown that the characteristic values of m are $K_t=\cos(\pi t/(T+1))$ and the characteristic values of $\bar{m}$ are $\bar{K}_t=\cos(2\pi t/T)$. The author obtains an approximation formula for the distribution of $\bar{r}=\bar{m}/p$ by replacing

the finite number of discrete values $\bar{R}_t$ by a continuous variable λ, distributed according to a density function suggested by, and as closely as possible approximating to, the scatter of the values $\bar{R}_t$. It is proposed to approximate the scatter of the values $\bar{R}_t$ by the density function $T/\pi(1-\lambda^2)^{\frac{1}{2}}$. The following approximation formula for the probability density of $\bar{r}$ is obtained:

$$(\tfrac{1}{2}T-1)2^{\frac{1}{2}T}\pi^{-1}\int_0^{\arccos \bar{r}}(\cos\alpha-\bar{r})^{\frac{1}{2}T-2}\sin\tfrac{1}{2}T\alpha\sin\alpha\,d\alpha.$$

By a similar method an approximation formula for the distribution of $r=m/p$ is also derived. *A. Wald.*

Fischer, Carl H. A sequence of discrete variables exhibiting correlation due to common elements. Ann. Math. Statistics 13, 97–101 (1942). [MF 6390]

In continuation of two earlier papers [Ann. Math. Statistics **4**, 103–126, 278–284 (1933)], the author treats bivariate distribution functions, regression lines and simple, multiple and partial correlation coefficients for pairs of variables in a set of k variables $x, y, \cdots$, formed in specified manner so as to introduce correlation. "The first, x, is equal to the number of white balls in a set of s_1 balls drawn one at a time from an urn which is so maintained that the probability of drawing a white ball is always a constant p. The second, y, is equal to the number of white balls in a second set of s_2 balls formed by drawing t_{12} balls at random from the s_1 balls of the first set plus s_2-t_{12} balls drawn directly from the urn." For the other drawings, similar restrictions are imposed. The regression equations are found to be linear. And r_{1k}, the coefficient of correlation between first and kth variables, is given by

$$r_{1k}=r_{12}\cdot r_{23}\cdot\ \cdots\ \cdot r_{k-1\,k}.$$

E. L. Dodd (Austin, Tex.).

Scheffé, Henry. An inverse problem in correlation theory. Amer. Math. Monthly **49**, 99–104 (1942). [MF 6242]

Given a square array of numbers r_{ij} for which $r_{ij}=r_{ji}$, $r_{ij}^2\leqq 1$, $r_{ii}=1$, how can one be sure of the existence of a set of data that would give rise to such hypothetical correlation coefficients r_{ij}? Under the restriction that the determinant of the array be not zero, the necessary and sufficient conditions are shown to be that certain $n-1$ determinants be all positive. The general case is also considered. No practical importance is attributed to this interesting mathematical problem. *W. A. Shewhart* (New York, N. Y.).

Shrivastava, M. P. On the D^2-statistic. Bull. Calcutta Math. Soc. **33**, 71–86 (1941). [MF 6171]

In this paper the author derives the exact distribution of the Mahalanobis D^2-statistic (originally determined by R. C. Bose [Sankhya **2**, 143–154 (1936)]) by means of characteristic functions. If k values of D^2 are obtained from k independent samples drawn from normal multivariate populations with the same population parameters, it is shown that the mean of the k values of D^2 is distributed according to a law of the same functional form as that for a single D^2 but with different values of the parameters involved. This additive property was originally established by R. C. Bose [Sankhya **3** (1938)]. The author gives the distribution functions for the difference and quotient of two values of D^2 drawn independently from the same D^2 distribution. *S. S. Wilks* (Princeton, N. J.).

Bhattacharyya, B. C. An alternative method of the distribution of Mahalanobis's D^2-statistic. Bull. Calcutta Math. Soc. **33**, 87–92 (1941). [MF 6172]

The exact distribution of the Mahalanobis D^2-statistic was first given by R. C. Bose [Sankhya **2**, 143–154 (1936)] by using a rather complicated transformation on the random variables involved. In the present paper the author obtains the distribution of the D^2-statistic by the usual Fourier inversion of the characteristic function. *S. S. Wilks.*

Merrington, Maxine. Table of percentage points of the t-distribution. Biometrika 32, 300 (1942). [MF 6530]

This table is derived from Miss Thompson's tables of percentage points of the incomplete beta function [Biometrika **32**, 168–181 (1941); these Rev. **3**, 153] and, in the notation of that paper, gives the values of

$$t=(p(1-x)/qx)^{\frac{1}{2}}$$

to 5 decimal places for $\nu_1=2q=1$, and for P and $\nu_2=2p$ over the same range of values as in Miss Thompson's table. *W. E. Milne* (Corvallis, Ore.).

Wilson, Edwin B. and Worcester, Jane. Note on the t-test. Proc. Nat. Acad. Sci. U.S.A. **28**, 297–301 (1942). [MF 6955]

In reference to the statement of R. A. Fisher that the function of the fiducial argument is to reject or fail to reject, at a stated level of probability, a hypothesis to which it is appropriate, the authors discuss the joint distributions of Student's t, the mean and the standard deviation of samples drawn from a normal population. Two diagrams represent overlapping regions in the fields of variation of these statistics where one of them would be judged significant. *J. Neyman* (Berkeley, Calif.).

Paulson, Edward. An approximate normalization of the analysis of variance distribution. Ann. Math. Statistics 13, 233–235 (1942). [MF 6924]

The statistic $F=s_1^2/s_2^2$ can be written

$$F=\frac{\chi_1^2}{n_1}\div\frac{\chi_2^2}{n_2}.$$

According to E. B. Wilson and M. M. Hilferty [Proc. Nat. Acad. Sci. U.S.A. **17**, 684–688 (1931)] the quantity $(\chi^2/n)^{\frac{1}{3}}$ is nearly normally distributed; hence $F^{\frac{1}{3}}$ is the ratio of two nearly normally distributed random variables. Combining this remark with a result of E. C. Fieller [Biometrika **24**, 428–440 (1932)], who showed that a certain function of the ratio of two normal variables is nearly normally distributed, the author obtains a function U of F such that U has a nearly normal distribution for $n_2\geqq 3$. Numerical illustrations for the goodness of the approximation are given. *Z. W. Birnbaum* (Seattle, Wash.).

Beall, Geoffrey. The transformation of data from entomological field experiments so that the analysis of variance becomes applicable. Biometrika 32, 243–262 (1942). [MF 6525]

The distributions observed in entomological experimentation usually exhibit a considerable correlation of the population S. D. and its mean, which makes it impossible to apply the analysis of variance to the treatment of such experiments. The author suggests that, if instead of the original observations x one would use their functions $y=k^{-\frac{1}{2}}\sinh^{-1}(kx)^{\frac{1}{2}}$, then the variance of y would be independent of its mean. *J. Neyman* (Berkeley, Calif.).

Satterthwaite, Franklin E. A generalized analysis of variance. Ann. Math. Statistics **13**, 34–41 (1942). [MF 6381]

The author sets down quite concisely the general and essential form of the analysis of variance with the necessary theory for its application not merely to the standard designs in existence but to any design to which this method applies. The application of the general technique as outlined is numerically illustrated by the complete solution of a well-chosen example. *C. C. Craig* (Ann Arbor, Mich.).

Cochran, W. G. Sampling theory when the sampling-units are of unequal sizes. J. Amer. Statist. Assoc. **37**, 199–212 (1942). [MF 6801]

The author points out that in certain sampling operations it is necessary to use sampling-units that differ in size. This paper is an expository discussion of the problem of how the size of the sampling-unit should be taken into account in selecting the sample and in making estimates from the results of the sample. The methods of estimation discussed are based on regression theory. *S. S. Wilks.*

Dieulefait, Carlos E. Note on a method of sampling. Ann. Math. Statistics **13**, 94–97 (1942). [MF 6389]

This author takes up the problem of Olds [Ann. Math. Statistics **11**, 355–358; these Rev. **2**, 112]: Given a lot of size $m=s+r$ containing s items of a specified kind. Items are drawn without replacement until j of the s items have been drawn. The problem is to determine the probability law of n, the number of drawings which have to be made. He obtains, using a different method, as a limiting form for the probability function of n (the size of a drawing $j\leqq n\leqq r+j$), a normal probability distribution. The reviewer takes this opportunity to retract any scepticism he may have suggested as to the novelty of any formula in Olds' original results. *A. A. Bennett* (Aberdeen, Md.).

Dwyer, P. S. Grouping methods. Ann. Math. Statistics **13**, 138–155 (1942). [MF 6918]

It is shown that, if at the time of forming a grouped frequency distribution of the observed values of a statistical variable the sums of the values in each class, as well as the numbers, are recorded, then one can, using both these items of information, calculate estimates of the second and higher order moments with decreased bias as compared with the results of the usual Shepard adjustments. If also the sums of squares by classes are recorded, still less biased estimates of the higher moments can be obtained. The theory is developed and numerically illustrated with a discussion of advantageous computing techniques. The analogous procedure in the case of product moments is sketched.

C. C. Craig (Ann Arbor, Mich.).

Stephan, Frederick F. An iterative method of adjusting sample frequency tables when expected marginal totals are known. Ann. Math. Statistics **13**, 166–178 (1942). [MF 6920]

In a previous paper W. E. Deming and the author [Ann. Math. Statistics **11**, 427–444 (1940); these Rev. **2**, 232] discussed the problem of adjusting the observed frequencies in a frequency table by least squares so that the marginal totals will agree with the known expected values. In the present paper the weights to be ascribed to the observed cell frequencies are supposed known and an iterative method of solving the normal equations is described and illustrated. The method is developed for the general case of minimizing the weighted sum of squares of such differences in which the frequencies are subject to a number of linear conditions not necessarily all independent. The reviewer was unable to arrive at the precise inequality (24) of the paper but this does not affect the method as it is to be applied. The results of the iteration process are shown to converge to the least squares solution. For a particular mode of application there is a discussion of the rapidity of the convergence. The reviewer was unable to follow the author's statement that any adjustment that satisfies the condition equations eliminates a portion of the errors of sampling and thus reduces χ^2. *C. C. Craig* (Ann Arbor, Mich.).

Anderson, Paul H. Distributions in stratified sampling. Ann. Math. Statistics **13**, 42–52 (1942). [MF 6382]

Let the continuous frequency function $f(x)$ be greater than zero for $a\leqq x\leqq b$ and zero elsewhere. Let $i-1$ values of x be so chosen as to divide the area under $f(x)$ into i subareas of size M_i. A stratified sample is defined as one consisting of i subsamples from the i subareas of size $m_i=kM_i$. Distributions of means of such stratified samples from rectangular, right triangular and normal frequency functions $f(x)$ are compared with corresponding distributions of means of random samples of equal size. Since the distributions of means of such stratified samples show in general less variability and sometimes less skewness than the corresponding ones for random samples, it is recommended that stratified samples be used where possible. It is not made clear, however, why one would want to take a sample of any kind if he already knows $f(x)$ well enough to break it up as directed. *W. A. Shewhart.*

Hasel, A. A. Estimation of volume in timber stands by strip sampling. Ann. Math. Statistics **13**, 179–206 (1942). [MF 6921]

Daniels, H. E. A method of improving certain routine measurements. Suppl. J. Roy. Statist. Soc. **7**, 146–150 (1941). [MF 6538]

Assume that interest is centered in some statistic calculated from the observed frequency distribution about which no prior knowledge is available and in which there are to be s cell divisions. After a certain number of measurements, it may become obvious that there is a preponderance of observed values falling within, let us say, the first few cells. It is shown that under certain specified conditions it may be more efficient to build up the tail of the distribution by successively omitting measurements in the first cells showing a preponderance of values rather than to record all the data. Having decided upon such a procedure, a method is given for determining the most efficient way of distributing a given number n of measurements between the successively truncated stages. Although some limitations of the method are pointed out, no mention is made of the loss of information that may result from truncation when the underlying chance cause system is not in a state of statistical control, as is usually the case.

W. A. Shewhart (New York, N. Y.).

Camp, Burton H. Some recent advances in mathematical statistics. I. Ann. Math. Statistics **13**, 62–73 (1942). [MF 6384]

A compact but detailed survey of recent progress in statistical theory exemplified in 51 cited articles most of

which have appeared in the last five years. These concern subject matter grouped under four main headings: (i) the theory of tests, (ii) estimation, (iii) likelihood tests, (iv) the method of randomization, each with subheadings. This timely exposition does not claim to present for the first time any new discoveries by the author. *A. A. Bennett.*

Craig, Cecil C. Recent advances in mathematical statistics. II. Ann. Math. Statistics 13, 74–85 (1942). [MF 6385]

The author treats discursively concerning a number of recent developments in statistical theory embraced in 30 listed recent papers: Hotelling's canonical variable, R. A. Fisher's linear discriminant functions, distributions of runs and gaps (as with Tippett's numbers) as investigated by Kermack, McKendrick, Mood, Wald and Wolfowitz, developments in the use of the probability integral transformation, leading to a study of test criteria as by E. S. Pearson and J. Neyman's smooth test of goodness of fit. *A. A. Bennett* (Aberdeen, Md.).

Aitken, A. C. and Silverstone, H. On the estimation of statistical parameters. Proc. Roy. Soc. Edinburgh. Sect. A. 61, 186–194 (1942). [MF 6455]

If $x_1, x_2, \cdots, x_n$ is a sample from a population obeying the probability function $\phi(x, \theta)$, it is required to estimate the unknown parameter θ from the sample. This paper gives results of postulating that the estimating t shall be unbiased and that $t-\theta$ shall have minimum variance. If Φ is the probability of obtaining the sample as a whole, then it is shown that the estimating function t exists provided that $\partial \log \Phi/\partial\theta$ has the form $(t-\theta)/\lambda(\theta)$. Furthermore, $\lambda(\theta)$ is the variance of t. Here nothing need be assumed about the form of distribution of $t-\theta$ and the results are valid for finite samples. The process of finding a t when one exists is equivalent to maximum likelihood estimation. The general form of $\phi(x, \theta)$ for which an estimating t exists under these postulates is found, and then it is noted that a result due to B. O. Koopman shows that such ϕ's admit sufficient statistics for the estimation of θ. An interesting side result is the fact that of the Pearson system of frequency functions only the normal law of error admits the mean as a sufficient statistic for locating the curve. *C. C. Craig.*

✱ **Wald, Abraham. On the Principles of Statistical Inference.** Notre Dame Mathematical Lectures, no. 1. University of Notre Dame, Notre Dame, Ind., 1942. 50 pp. $1.00.

The lithoprinted booklet is divided into six chapters: (1) Introduction; (2) The Neyman-Pearson theory of testing statistical hypotheses; (3) R. A. Fisher's theory of estimation; (4) The theory of confidence intervals; (5) Asymptotically most powerful tests and asymptotically shortest confidence intervals; (6) Outline of a general theory of statistical inference. The last two chapters are devoted to the author's own recent results. In the others he deals with such notions of modern statistical theory as are necessary for the understanding of his chapters 5 and 6. There is a good deal of material and the booklet would be interesting reading to everyone who would like to have a condensed review of the basic conceptions of mathematical statistics. The booklet is written with the obvious intention of clarifying these conceptions and, while the author insists on definitions and describes the known results relating to them, no detailed proofs are given. *J. Neyman.*

Wald, Abraham. Asymptotically shortest confidence intervals. Ann. Math. Statistics 13, 127–137 (1942). [MF 6917]

Denote by E_n the set of n mutually independent random variables each following the same elementary probability law $f(x|\theta)$ depending on just one unknown parameter θ. Let further $\delta_n(E_n)$ denote an interval function of E_n. Generalizing the conception of the shortest system of confidence intervals and that of the short unbiased system, the author gives the following definitions: (1) A sequence of interval functions $\{\delta_n(E_n)\}$, for $n=1, 2, \cdots$, is called an asymptotically shortest confidence interval of θ if it satisfies the following conditions: (a) $P\{\delta_n(E_n)c\theta|\theta\}\equiv\alpha$ identically in θ; (b) for any other sequence of interval functions $\{\delta_n'(E_n)\}$ which satisfies (a), the least upper bound of $P\{\delta_n(E_n)c\theta'|\theta''\} - P\{\delta_n'(E_n)c\theta'|\theta''\}$, taken with respect to the variation of θ' and θ'', tends to zero as $n\rightarrow\infty$. (2) A sequence of interval functions $\{\delta_n(E_n)\}$ is called an asymptotically shortest unbiased confidence interval of θ if it satisfies the condition (a) above and also the following conditions (b′) and (c′): (b′) the least upper bound of $P\{\delta_n(E_n)c\theta'|\theta''\}$ taken with respect to θ' and θ'' tends to α as $n\rightarrow\infty$; (c′) for any sequence of interval functions $\{\delta_n'(E)\}$ satisfying (a) and (b′), the least upper bound of $P\{\delta_n(E_n)c\theta'|\theta''\}-P\{\delta_n'(E_n)c\theta'|\theta''\}$ tends to zero as $n\rightarrow\infty$. The author proves a number of interesting theorems concerning the above conceptions, which, however, are difficult to present in a short abstract. The last section of the paper establishes the connection between the conceptions of the author and those of confidence intervals, shortest in the average, previously introduced by Wilks [Ann. Math. Statistics 9, 166–175 (1938)]. *J. Neyman*

Wald, Abraham. Some examples of asymptotically most powerful tests. Ann. Math. Statistics 12, 396–408 (1941). [MF 6050]

In an earlier paper [Ann. Math. Statistics 12, 1–19 (1941); these Rev. 3, 8] Wald has given the definition of an asymptotically most powerful test and has shown that many of the commonly used tests based on the maximum likelihood estimate are asymptotically most powerful. In the present paper further examples of asymptotically most powerful tests are given. More specifically, suppose $f(x, \theta)$ is the probability density function for the given population, and for a given sample of n elements let

$$y_n(\theta)=n^{-1}\sum_1^n \partial \log f(x_\alpha, \theta)/\partial\theta.$$

Let W_n' be the region in sample space for which $y_n(\theta_0)\geqq c_n'$, W_n'' that for which $y_n(\theta_0)\leqq c_n''$ and W_n that for which $|y_n(\theta_0)|\geqq c_n$, where c_n', c_n'', c_n are chosen so that the probability is α of a sample point falling into each of the regions W_n', W_n'', W_n for $\theta=\theta_0$. Wald shows that under certain reasonable restrictions on $f(x, \theta)$ the sequence $\{W_n'\}$ is an asymptotically most powerful test of the hypothesis $\theta=\theta_0$, if θ is restricted to the values $\theta\geqq\theta_0$. Similarly, $\{W_n''\}$ is an asymptotically most powerful test of the hypothesis $\theta=\theta_0$ if θ is restricted to the values $\theta\leqq\theta_0$, while $\{W_n\}$ is an asymptotically most powerful test of $\theta=\theta_0$ if θ can take on any real value. It is shown that the Neyman-Pearson type A critical region is an asymptotically most powerful unbiased test of the hypothesis $\theta=\theta_0$. *S. S. Wilks.*

Kolmogoroff, A. Confidence limits for an unknown distribution function. Ann. Math. Statistics 12, 461–463 (1941). [MF 6056]

Wald and Wolfowitz [Ann. Math. Statistics 10, 105–118 (1939)] have shown how to construct confidence limits for

a continuous cumulative probability function $F(x)$ from a sample of n elements from the distribution. In the present note Kolmogoroff simply states four theorems, without proof, which he published previously [Giorn. Ist. Ital. Attuari 4, 83–91 (1933)] and which are related to the Wald-Wolfowitz problem to some extent. *S. S. Wilks.*

Pearson, E. S. Notes on testing statistical hypotheses. Biometrika 32, 311–316 (1942). [MF 6533]

The author's contribution to the discussion that originated at the International Conference on the Applications of the Calculus of Probability, Geneva, 1939. [See also the following review.] The main points discussed are: (a) In choosing a test for a statistical hypothesis, is it possible or even necessary to specify the hypotheses alternative to that tested? Why should not a test be made to depend only on the form of law associated with the hypothesis tested? (b) Is the method of approach to these problems, as represented by the theory of testing statistical hypotheses, applicable to testing the appropriateness of probability laws or only to testing hypotheses regarding numerical values of constants contained in these laws? Apart from answers to question (a) as given by Pearson, one should perhaps mention that the principle that the device of criteria should be made solely on the basis of hypothesis tested is insufficient. In fact, it is possible to define two criteria which, on the hypothesis tested, have the same distribution so that the above principle would not allow choosing one of them. On the other hand, these criteria have the property that, while one indicates most strongly that the hypothesis tested is wrong, the other is bound to affirm the opposite. [See reviewer's "Lectures and Conferences on Mathematical Statistics," Washington, D. C., 1937, Lecture 3.]

J. Neyman (Berkeley, Calif.).

Gumbel, E. J. Simple tests for given hypotheses. Biometrika 32, 317–333 (1942). [MF 6534]

This is another study of the use of the probability integral transformation to test the hypothesis that an observed distribution of values of a single variable is a random sample from a given population. One innovation is the suggested use of control curves. These are obtained by adding to or subtracting from each calculated ordinate the standard error of that ordinate. One may prefer the one of two comparable hypotheses for which the more points fall within the control curves. If the range for the variable be divided into uniform intervals, on the application of the probability integral transformation for a given hypothesis to the observed frequencies observed points obtained may be compared with the uniform distribution on the hypothesis. The usual tests such as χ^2 or the likelihood ratio may be applied. But it may happen that two different hypotheses may not be distinguished if only the numbers of points in each interval be counted. In this case the author develops a test in which the actual position of each point is compared with its mean position on the hypothesis tested, the test being similar in form to the ω^2-test of Cramér and von Mises.

C. C. Craig (Ann Arbor, Mich.).

Tsao, Fei. Tests of statistical hypotheses in the case of unequal or disproportionate numbers of observations in the subclasses. Psychometrika 7, 195–212 (1942). [MF 7140]

Methods for effecting the analysis of variance in the case of unequal class frequencies are available in papers of Yates and Wilks. Using the method of fitting constants due to Yates, the present author develops the explicit equations for the various component sums of squares which will be useful for workers who find it difficult or tedious to apply the general methods. *C. C. Craig* (Ann Arbor, Mich.).

Haldane, J. B. S. The fitting of binomial distributions. Ann. Eugenics 11, 179–181 (1941). [MF 5925]

Some frequency distributions can be well fitted by binomial distributions $P_r=\binom{k}{r}p^r(1-p)^{k-r}$ with negative p and k. To estimate the parameters p and k the first two moments of the observed frequency distribution may be used. H. Jeffreys pointed out [Theory of Probability, Clarendon Press, Oxford, 1939, pp. 259–260; cf. these Rev. 1, 151] that this method is not efficient, and suggested that the fitting may be done by the method of maximum likelihood, which seemed to require the use of tables of digamma functions. In the present paper it is shown that the use of those tables may be avoided and that elementary operations only are needed in estimating p and k by the method of maximum likelihood. *Z. W. Birnbaum* (Seattle, Wash.).

Fisher, R. A. The negative binomial distribution. Ann. Eugenics 11, 182–187 (1941). [MF 5926]

The theoretical efficiency of estimating the parameters p, k of a negative binomial distribution [see the preceding review] by the first two moments of the observed distribution is computed. It is shown that in some cases this efficiency is good enough for practical purposes.

Z. W. Birnbaum (Seattle, Wash.).

Fisher, R. A. The likelihood solution of a problem in compounded probabilities. Ann. Eugenics 11, 306–307 (1942). [MF 7077]

Consider $s+1$ sequences of independent trials. Let p_i denote the probability of a success in each of the n_i trials, forming the ith sequence, $i=1, 2, \cdots, s$. Assume further that the probability of a success in the $(s+1)$st sequence is known to be equal to $\prod_{i=1}^{s} p_i$. The author deduces formulae giving the maximum likelihood estimates of the probabilities p_i, based on the observable numbers of successes in each of the $s+1$ series of trials. *J. Neyman.*

Wilson, Edwin B. The controlled experiment and the four-fold table. Science (N.S.) 93, 557–560 (1941). [MF 6454]

Fisher, R. A. The interpretation of experimental four-fold tables. Science (N.S.) 94, 210–211 (1941). [MF 6453]

Bonnier, Gert. The four-fold table and the heterogeneity test. Science (N.S.) 96, 13–14 (1942). [MF 6883]

Wilson, Edwin B. On contingency tables. Proc. Nat. Acad. Sci. U.S.A. 28, 94–100 (1942). [MF 6330]

This is a controversy concerning the appropriate test of the hypothesis that the probabilities of a certain event in two series of trials are the same. While Fisher favors the χ^2 test, the two other participants in the discussion raise doubts. *J. Neyman* (Berkeley, Calif.).

Wilson, Edwin B. On confidence intervals. Proc. Nat. Acad. Sci. U.S.A. 28, 88–93 (1942). [MF 6329]

Denote by p the probability of a particular outcome E of a random trial and by p_0 the relative frequency of E in n such independent trials. Using as an illustration the problem of estimating p when p_0 and n are known, the author emphasizes the distinction between the concept of confidence intervals and his own method of approach as explained in

his earlier paper [J. Amer. Statist. Assoc. 22, 209–212 (1927)]. The author is doubtful about the soundness of the former and suspects that the consideration of confidence intervals leads to that of negative probabilities. While this criticism does not seem to be justified, the reviewer readily agrees that the statement of the problem of confidence intervals is different from that treated by E. B. Wilson in his paper quoted above. *J. Neyman* (Berkeley, Calif.).

v. Mises, R. On the correct use of Bayes' formula. Ann. Math. Statistics 13, 156–165 (1942). [MF 6919]

The problem actually discussed is that of statistical treatment of bacteriological tests of water supplies. If n samples of water are analyzed and x of them found infected the author suggests that the lower bound of the posterior probability of the average number λ of bacteria per sample be calculated, λ being between zero and one. The estimate of this lower bound is obtained from Bayes' formula with properly estimated extremes of the a priori probability distribution of λ. The latter is estimated from past experience in analyzing water. The final conclusion reached by the author with respect to a particular set of data is as follows: if we assume that in continuing the experiments the distribution of test results [presumably of λ (Reviewer)] will be about the same as it has been in the past 3420 cases, we have a chance of more than 99.9% of being right, when we state in each case of no positive test (out of a set of 5) that the density of bacteria is less than 1 per 10 cc. While the above statement is perfectly correct, the reviewer is inclined to doubt whether it represents a solution of the practical problem considered. The fact that the bacteriological tests of water are continued incessantly and, undoubtedly, would be continued even if a few thousand of them in a row did not show any single infected sample, implies that it is expected that the past distribution of λ need not necessarily be again observed in the future. The statistical problem that meets the practical situation seems to be a little different from that treated by the author. If k denotes the concentration of bacteria which is satisfactory, and $l>k$ the one which is considered already dangerous, the statistician may be asked to determine the cheapest way of handling the analysis so that (i) if $\lambda\leqq k$, then some 99% of sets of samples would be found as satisfactory, and (ii) if $\lambda\geqq l$, then at least 99.9% of sets of samples would indicate that the water is bad. Thus, as far as the practical side of the situation is concerned, it presents a problem of testing the hypothesis that $\lambda\geqq l$. This presumption of the reviewer does not agree with the author's last section suggesting that the statistical problem involved is that of estimation and might be treated by confidence intervals. *J. Neyman.*

Glock, Waldo S. A rapid method of correlation for continuous time series. Amer. J. Sci. 240, 437–442 (1942). [MF 6671]

In place of the Pearsonian coefficient of correlation, the author suggested [Climatological Researches, as reported by A. E. Douglas, Carnegie Inst. Washington, Yearbook No. 32, 208–211 (1933), in particular, p. 209] measures of correlation which he now regards as having given satisfactory results after eight years of use in comparisons of rainfall with tree growth. The method employed avoids calculation of trend lines and mean lines. It is, indeed, a phase of the variate difference method which studies the nature of sequences by examination of the differences of consecutive terms. Let X designate the terms of a sequence of first differences of one variate, and Y those of another. Of the products XY, let m products be positive, with sum P, and n be negative with sum $-N$. The author defines: $t=(P-N)/(P+N)$; $i=Pn/Nm$; $T=t/i$. To measure correlation, t is used if i is close to unity, otherwise T. In this paper these measures are not associated with probability, but it is obvious that $|t|\leqq 1$, and that if the original variates, from which the differences X and Y are taken, move at each step in the same direction, then $P=1$; if in the opposite direction, $P=-1$. In conclusion, the author states: "As in other types of correlation, a high coefficient (say of +0.80 or +0.90) suggests but does not prove relationship." Certain plots of tree growth and rainfall are given, and the computations made. *E. L. Dodd* (Austin, Tex.).

Schumann, T. E. W. On Yule's method of investigating periodicities of disturbed series. The motion of a pendulum in a turbulent fluid. Philos. Mag. (7) 33, 138–150 (1942). [MF 6258]

The author discusses the motion of a damped pendulum induced by (auto-correlated) random disturbances. The first such study was made by Yule [Philos. Trans. Roy. Soc. London. Ser. A. 226, 267–298 (1927)]. Schumann finds the correlation function of the pendulum's angular displacement y, that is, the expectation of (y at time t)·(y at time $t+s$), in terms of the correlation function of the disturbances. The work does not satisfy the present-day standards of precision; for example, the formulations are not made so carefully that there is any distinction between sample and population averages. As an application, an experimental set-up is described for measuring wind turbulence in terms of its effect on a pendulum. *J. L. Doob.*

Fisher, R. A. New cyclic solutions to problems in incomplete blocks. Ann. Eugenics 11, 290–299 (1942). [MF 7075]

This paper deals with special solutions of the problem of arranging s^2+s+1 objects (or varieties), each taken $s+1$ times, in s^2+s+1 blocks of $s+1$, in such a way that every two blocks contain just one common member. The solutions given for $s=8$ and $s=9$ are believed to be new; but the former is identical with J. Singer's [Trans. Amer. Math. Soc. 43, 377–385 (1938), in particular, p. 384], and the latter is quite similar. In the case $s=9$, the 91 varieties are denoted by the residue-classes modulo 91; the first block is (1, 2, 7, 11, 24, 27, 35, 42, 54, 56), and the others are derived by adding 1 throughout, then 2, and so on. It appears that much labor could have been saved by regarding the varieties and blocks as points and lines of a finite projective geometry. We easily verify that three collinear points have a unique fourth harmonic point (for example, 56 is the harmonic conjugate of 7 with respect to 1 and 2); therefore the geometry is in fact $PG(2, 3^2)$. The derived Latin squares are all of one species because of the fundamental theorem of projective geometry, but the Eulerian (or Graeco-Latin) squares are of two species as four collinear points may or may not be harmonic. *H. S. M. Coxeter* (Toronto, Ont.).

Applications of the Theory of Probability

Wright, Sewall. Statistical genetics and evolution. Bull. Amer. Math. Soc. 48, 223–246 (1942). [MF 6402]

This sixteenth Gibbs lecture is a review of the prominent work done by the author in the last twelve years towards

the establishment of a mathematical theory of evolution. Hardy's formula, a mathematical consequence of Mendel's inheritance mechanism, requires the persistency of the relative frequency of genes in each set of alleles of an indefinitely large random breeding population free from evolutionary agents. Therefore the change of this frequency (called briefly the gene frequency q) is a measure of the evolution of the population. As a consequence of biologically plausible hypotheses compatible with the requirements of mathematical simplicity the change of q per generation is, in the case of an indefinitely large random breeding diploid population,

$$\Delta q = -uq + v(1-q) - m(q-q_i) + \tfrac{1}{2}q(1-q)\partial \log W/\partial q,$$

where each of the four terms is the contribution of a single evolutionary agent; u and v are the mutation intensities (v representing the reverse mutation), m the intensity of the genetic exchange with an immigrant population whose gene frequency is q_i. The last term is the effect of the selection; W is a polynomial of second degree in q containing three coefficients which represent the relative selective advantages of the paired alleles. In a population of finite size there is, besides Δq, a random variation of gene frequency δq, due to the statistic character of the Mendelian mechanism; the corresponding decrease of heterozygosis is inversely proportional to the number N of breeding individuals of the population.

Introducing the approximation $(\Delta q)^2 \approx 0$, the author succeeds in deriving an explicit expression of the distribution function $\varphi(q)$ of the gene frequencies satisfying the conditions of a statistical equilibrium (constant mean and constant variance). The distribution curve is U-shaped in small populations and I-shaped in large populations. It shows clearly the slight effect of selection in small populations. The author considers next the distribution function $\varphi(q)$ under the condition that the fixation of one of the alleles (that is, its reaching the frequency $q=1$) is an irreversible process. The intensity of this process (fixation rate K) is assumed constant and is found to be equal to $\varphi(1-\frac{1}{2}N^{-1})/4N$. From the consideration of the mean and of the variance the author obtains two integral equations for $\varphi(q)$. He states that he has not been able to find a general expression of $\varphi(q)$ satisfying these equations. However a more direct analysis of the process gives him the following equation (which he solves in some important cases):

$$(1-K)\varphi(Q) = F(Q, N)\int_0^1 (q+\Delta q)^{2NQ} \times (1-q-\Delta q)^{2N(1-Q)}\varphi(q)dq,$$

where $F(Q, N)$ is a known function of Q and N.

The paper contains also indications concerning more general cases (polyploidy, sex linkage, inbreeding, etc.) and closes with general considerations on the evolution of a population and its separation into new species, as suggested by the mathematical theory. *I. Opatowski.*

Shock, Nathan W. and Morales, Manuel F. A fundamental form for the differential equation of colonial and organism growth. Bull. Math. Biophys. **4**, 63–71 (1942). [MF 6678]

Generalizing the so-called logistic law of growth, the authors assume that the growth is described by an equation of the form $dN/dt = A(t, N)N + B(t, N)N^k$; here N denotes the number of cells and it is assumed that the term AN arises from conditions within the growing cell itself, while BN^k arises from interactions between the growing cells of the community. Several special cases are treated.

W. Feller (Providence, R. I.).

Lotka, Alfred J. The progeny of an entire population. Ann. Math. Statistics **13**, 115–126 (1942). [MF 6916]

This paper discusses the solution of an integral equation in renewal theory. Physical details are considered and a numerical example is carried out in detail. *A. E. Heins.*

Anderson, R. D. On the application of quantum mechanics to mortality tables. J. Inst. Actuar. **71**, 228–258 (1942). [MF 6522]

The author gives an exposition of some elementary methods of quantum mechanics, in particular, of the use of matrices in general vector analysis. An analogy with elementary probabilities is established by using vectors to represent the composition of balls in a bag. Consider n mutually orthogonal unit vectors and let each represent a color. Different vectors of the same length will represent different allotments of the same number of balls to the n colors. Splitting a given vector into its components corresponds to a division of the balls into lots each containing one color only. All possible (not necessarily rectangular) coordinate systems correspond to the different possible divisions of the balls into lots. Similar ideas are used in connection with mortality tables, in particular, to deduce a new law of mortality. For details we must refer to the paper. *W. Feller* (Providence, R. I.).

Campbell, N. R. The replacement of perishable members of a continually operating system. Suppl. J. Roy. Statist. Soc. **7**, 110–130 (1941). [MF 6537]

Although the problem treated in this paper is seemingly very specialized, it is typical for a class of problems connected with the theory of self-renewing aggregates. "A series of street-lighting posts has to be kept supplied with lamps. Is it cheaper to replace the lamps individually as they fail, or to replace all the lamps (or at least all of a certain age) at regular intervals, replacing only those that fail between these intervals?" Part I treats this problem exhaustively from a theoretical point of view; in part II computations are performed for an empirical example; part III attempts to justify, by examples, the assumption that the density of replacements tends to an asymptotic limit [for a proof, cf. Feller, Ann. Math. Statistics **12**, 243–267 (1941); these Rev. **3**, 151]. *W. Feller.*

Rosenblatt, Alfred. Sur les théorèmes des petits nombres de Poisson, de Bortkiewicz et G. Pólya. Application aux phénomènes rares. I. Propagation des maladies contagieuses: peste bubonique au Brésil. Actas Acad. Ci. Lima **3**, 160–167 (1940). [MF 7111]

The Pólya-Eggenberger generalization of the Poisson law "of small numbers" is fitted to the incidence of bubonic plague in Sao Paulo, Brazil. The fit is found to be poor (without the use of tests of significance). *J. L. Doob.*

Chambers, E. G. and Yule, G. Udny. Theory and observation in the investigation of accident causation. Suppl. J. Roy. Statist. Soc. **7**, 89–101; discussion, 101–109 (1941). [MF 6536]

It is assumed that the number of accidents A happening to a person during a given period of exposure to risk has a Poisson distribution. The mean value λ of the distribution is called the degree of proneness of this individual. If λ is a

variable, that is, it varies from individual to individual, then the distribution of A in a large group of individuals is a compound Poisson distribution. The mean value $E(A)$ of A is equal to the mean value $E(\lambda)$ of λ and the variance $\sigma^2(A)$ of A is equal to $E(A)+\sigma^2(\lambda)$. It is shown that, if A and λ refer to a unit time of exposure and if we denote by A_k and λ_k the corresponding quantities for k time units of exposure, then $\sigma(\lambda_k)=k\sigma(\lambda)$, $E(A_k)=kE(A)$ and $\sigma^2(A_k)=kE(A)+k^2\sigma^2(\lambda)$. G. U. Yule considers the case where the value λ of any individual remains unchanged throughout the whole observation period and derives some expressions which are time invariants, that is, they do not depend on the length of time exposure. These theoretical results of Yule are applied by E. G. Chambers to some observed data. Chambers finds that in the light of the observed data the assumption of the constancy of λ of an individual throughout the observation period can hardly be maintained. A discussion of this paper by J. O. Irwin and M. Greenwood is included. *A. Wald* (New York, N. Y.).

Gumbel, E. J. On the frequency distribution of extreme values in meteorological data. Bull. Amer. Meteorol. Soc. **23**, 95–105 (1942). [MF 6637]

The author has made numerous contributions to the theory of extreme values. In the present paper, he applies his theory to: (1) precipitation in Boston, 1818–1940; (2) maximum temperature in Parc Saint-Maur, 1851–1939; (3) atmospheric pressure at Blue Hill Observatory, 1886–1940; (4) minimum temperature in Boston, 1872–1940. The 123 numbers giving the largest of the monthly rainfalls for each year were arranged in the order of their magnitude. The 82nd of these numbers is 7.89 inches. Then in one-third of the years there was precipitation in some month in excess of 7.89 inches. Thus three years is set down as the empirical "return period" for a monthly precipitation of more than 7.89 inches. For the theoretical "return period" and for certain other parameters, the author has developed formulas. He uses four different tests. In conclusion, the author states: "The four methods of comparison lead to a satisfactory fit between theory and observation. The goodness of fit diminishes of course with the number of observations. . . ." *E. L. Dodd* (Austin, Tex.).

Gumbel, E. J. Probability-interpretation of the observed return-periods of floods. Trans. Amer. Geophys. Union 1941, 836–850 (1941). [MF 6804]

Kimball, Bradford F. Limited type of primary probability distribution applied to annual maximum flood flows. Ann. Math. Statistics **13**, 318–325 (1942). [MF 7241]

MATHEMATICAL PHYSICS

***Kron, Gabriel. A Short Course in Tensor Analysis for Electrical Engineers.** John Wiley and Sons, Inc., New York, 1942. xv+250 pp. $4.50.

Scheffers, H. Bemerkungen zur allgemeinen Schwankungstheorie. Z. Phys. **117**, 444–451 (1941). [MF 6846]

The author considers the fluctuations of the generalized parameters $\alpha_1, \alpha_2, \cdots$, which together with the temperature T determine the state of a system. These parameters are associated with generalized external forces $A_1, A_2, \cdots$, such that $A_1d\alpha_1+A_2d\alpha_2+\cdots$ is the external work done by the system when the α's are changed slightly; this sum is a complete differential for a reversible isothermal change. It is shown that $\overline{\alpha^2}=-kT/(\partial A/\partial\alpha)_{T,\alpha=0}$, a result that includes the known theorem that the mean deviation-work is always $\frac{1}{2}kT$; it shows that the fluctuation of a parameter α is greater the less the stability of the system relative to this parameter. Various known fluctuation expressions are special cases of this formula. The author also gives a formula for the fluctuation $\overline{\Delta\omega^2}$ of any function ω of T and the parameters α; a special case of this formula, corresponding to $\alpha=v$, $A=p$, was given by Lorentz. The paper concludes with a discussion of fluctuations about a phase-equilibrium of the third kind [Justi and von Laue, S.-B. Preuss. Akad. Wiss. **17**, 237–249 (1934); Z. Tech. Phys. **15**, 521–529 (1934)], and at the critical point of a system. *S. Chapman.*

Kohler, Max. Schallabsorption in Mischungen einatomiger Gase. Ann. Physik (5) **39**, 209–225 (1941). [MF 6848]

The Chapman-Enskog development of the theory of the mean-free-path phenomena of a gas is applied to determine the effect of diffusion on the propagation of sound in a mixed monatomic gas, thus extending the discussions by Stokes [Trans. Cambridge Philos. Soc. **9**, 8–106 (1851)] and Kirchhoff [Ann. Physik (5) **14**, 177–193 (1868)] of the influence of viscosity and thermal conduction on sound propagation (in a simple gas). The formal treatment is detailed, and the results are discussed in their qualitative aspects, but there is little numerical illustration of them. The author refers to a discussion of the same subject by Y. Rocard [J. Phys. Radium (7) **1**, 426–437 (1930)], and points out that certain necessary factors are missing from Rocard's results, which invalidate, inter alia, the conclusion by Rocard that in air the influence of diffusion on the absorption of sound is similar in magnitude to that of thermal conduction. The author finds that, on the contrary, diffusion is far less important, thus agreeing substantially with the result of a much earlier simple discussion of sound propagation in air by S. Chapman and G. H. Livens [Proc. London Math. Soc. (2) **19**, 341–349 (1920)].

The author indicates that in principle it is possible through measurements of the absorption of sound in a mixed gas to determine the coefficient of thermal diffusion, "otherwise determinable only with difficulty." Actually this coefficient is not specially difficult to determine by direct measurement, but would be very difficult to measure accurately in the new way proposed, because diffusion is so minor a factor in sound-absorption. *S. Chapman.*

Fuchs, Klaus. Statistical mechanics of binary systems. Proc. Roy. Soc. London. Ser. A. **179**, 340–361 (1942). [MF 6238]

The partition function for a binary solid solution is evaluated in terms of cluster sums according to Mayer's method for gases, which can be adapted with slight modifications to the present problem. The expansion proceeds here in powers of the atomic fraction and singularities in this expansion correspond to phase transitions. They can be calculated in the case of a binary solution with a two phase region. The limits of solubility and the specific heat are determined. Results of numerical calculations are given for a body centered cubic lattice under the assumption

that the interaction between like and unlike atoms differ only if the two particles are nearest neighbors.

L. W. Nordheim (Durham, N. C.).

Fuchs, K. The statistical mechanics of many component gases. Proc. Roy. Soc. London. Ser. A. **179**, 408–432 (1942). [MF 6400]

The method of Mayer for the treatment of imperfect gases is applied to the mixture of an arbitrary number of constituents. The essential new step in this generalization consists in the proper reduction of the cluster integrals, which is quite different from the simpler cases treated before. It is possible to treat the problem of chemical reactions for imperfect gases with the new methods. Strict laws for the chemical equilibrium are derived. They reduce for sufficiently small pressures to the laws of mass action.

L. W. Nordheim (Durham, N. C.).

Chapman, Sydney and Majid Mian, A. The rate of ion-production at any height in the earth's atmosphere. Terr. Magnetism **47**, 31–44 (1942). [MF 6456]

The rate of ion production at height z in the atmosphere is expressed in terms of the maximum value I_0, when the curvature of the level layers of the atmosphere are neglected, the zenith distance χ and a quantity R related to the radius of the earth by the equation $HR=a+h_0$, where h_0 is the datum level in height-units H and the height in the same units is $h=h_0+Hz$. The expansion of I in a series of Legendre polynomials is

$$I/eI_0=\sum_{n=0}^{\infty} \alpha_n(R, z)P_n(\cos\chi),$$

and, when $R=\infty$, $p=e$,

$$\alpha_n(\infty, z)=(n+\tfrac{1}{2})p\int_1^{\infty} e^{pz}P_n(1/x)x^{-2}dx.$$

This integral is expressed as a linear combination of the integrals

$$E_m(p)=\int_1^{\infty} e^{-px}x^{-m}dx, \qquad m\geqq 2,$$

which are represented approximately by index sums

$$E_m'(x)=\sum e_s\epsilon_s^{2-m}e^{-\epsilon_s x}.$$

Numerical values of e_s and ϵ_s to 5 significant figures for $s=1$ to 5 are quoted from a former paper [Philos. Mag. (7) **33**, 115–130 (1942); cf. these Rev. **3**, 276] and are used to calculate the quantities $\alpha_n(\infty, z)$. A table of $10^3(e^p/p)\alpha_n(\infty, z)$ is given for $p=0(.1)1(.5)6(1)10$, $n=0(1)6$, the number of significant figures ranging from 1 to 4. The Legendre expansion is also checked numerically.

The curvature of the level layers of the atmosphere is next considered and $a_n(R, z)$ is expressed as an integral with respect to x, but the upper limit is a quantity X determined by the value χ_0 of χ for which $\sin\chi_0=a/(a+h)$. Calculations for this case are still to be made. The daily variation of the rate of ion production at any point is, however, expressed as a Fourier series and at the equinoxes the computations depend on the integrals

$$B_s(q)=(2/\pi)\int_0^{\pi/2} \exp(-q\sec\phi)\cos(s\phi)d\phi,$$

which are approximated by means of index sums. A table is given for $q=0(.2)1(1)20$ and $m=0(1)4$ of the quantity $10^3\cdot e^qB_m(q)$.

H. Bateman (Pasadena, Calif.).

Iyengar, K. S. K. Exact solution of the equations of the general cascade theory with collision loss. Proc. Indian Acad. Sci., Sect. A. **15**, 195–229 (1942). [MF 7116]

A rigorous solution is found for the general problem of cascade theory in which use is made of the exact Bethe-Heitler cross-sections for pair creation, and losses by radiation and collision are also taken into consideration. A solution satisfying the boundary conditions for the case of complete screening with collision loss was found by Bhabha and Chakrabarti by a formal procedure. A rigorous investigation is now given showing not only its correctness but its great value for numerical work with an approximation in which only the first term is retained. The analysis is hard to summarize briefly. Like previous authors, Iyengar makes much use of series and the Mellin transformation in the solution of the integrodifferential equations of the cascade theory for given boundary conditions. *H. Bateman.*

Somenzi, Vittorio. Interazione elettrodinamica di due elettroni e teoria di Welker della superconduttività. Nuovo Cimento (N.S.) **18**, 223–234 (1941). [MF 6676]

In 1938 H. Welker, after calculating the total diamagnetic moment of an electronic gas enclosed in a cylinder and finding it to be practically negligible in agreement with L. Landau's general prediction, advanced the hypothesis that the gap in the spectrum of the energy of the electron may be due to the interaction of exchange of the electron with the current to which it gives rise. It was indeed surmised that this interaction is not always negligible in comparison with the classical electrodynamic interaction, as it is in the case of the free electron. To test this hypothesis Welker considered models of the type used by Sommerfeld and Bethe, in which the electrons and currents are more or less localized near the nodes of a lattice.

Somenzi now shows that the two interactions are also of the same order of magnitude in the case of the models used by Bloch, Peierls, Wigner and Seitz, Heitler, London and Heisenberg. The electrodynamic interaction of two electrons is first calculated. A comparison is then made with the interaction of exchange between two electrons l-s of the same atom. The electrodynamic interaction of two electrons in a metal is then found. Bloch's model of strongly bound electrons is treated by considering the case of a linear chain of atoms. The analysis for the model of Wigner-Seitz is briefly sketched. A possibility, proposed by Gentile, is then mentioned. If the N atoms of a crystal are divided into m groups each containing N/m atoms the electrostatic interaction between these elementary domains is reduced in the ratio $1/m$ and so becomes comparable with the electrodynamic effect which tends to act between the electrons. Similarly the electrodynamic interaction should be due in a certain sense to a localization either of the electrons or of the currents.

H. Bateman (Pasadena, Calif.).

Pincherle, L. Eigenfunctions in Heisenberg approximation of the two-electrons problem. Philos. Mag. (7) **33**, 462–466 (1942). [MF 7072]

The Heisenberg method for the two-electron problem involves the use of an approximate potential energy term which, as a function of the radial co-ordinate r, changes its form abruptly at a certain value r_0, which is a sort of average distance of the two electrons from the nucleus. This approximation was previously used in cases where one electron is almost constantly much nearer to the nucleus than the other. The author obtains analytic expressions for the eigen-

functions of the radial equation with this potential, and shows that, even when the two electrons have the same average nuclear distance, the resulting values for the energy terms check well with experimental values, and with the results of more complicated, and presumably more accurate, approximate methods. For values of r less than r_0, the expressions found for the eigenfunctions involve a hypergeometric series. For $r>r_0$, this series is replaced by an asymptotic series in $1/r$. The error necessarily involved in the use of asymptotic series turns out to be not too large. One advantage of the explicit formulas obtained for the eigenfunctions is that all approximations of the perturbation theory may be actually carried out without tedious numerical integrations. *O. Frink* (State College, Pa.).

Madhava Rao, B. S. Commutation rules for matrices related to particles of higher spins. Proc. Indian Acad. Sci., Sect. A. 15, 139–147 (1942). [MF 7040]

For a wave equation of the form $\partial_\mu \beta_\mu \psi + \chi\psi = 0$, analogous to the Dirac equation of the electron, the author determines the proper form of the commutation rules for the matrices β_μ in the case of particles of spin $\frac{3}{2}$ and 2. Three assumptions are made: (1) that the wave equation is relativistic invariant, (2) that the spin operator has the form $k(\beta_\mu\beta_\nu - \beta_\nu\beta_\mu)$, where k is a numerical constant, and (3) that the eigenvalues of any component of the spin operator are $-\frac{3}{2}$, $-\frac{1}{2}$, $\frac{1}{2}$, $\frac{3}{2}$ for the case of spin $\frac{3}{2}$, and $-2, -1, 0, 1, 2$, for the case of spin 2. Various consequences of these assumptions are condensed into a single complicated commutation rule [the author's equation (26)] for the case of spin $\frac{3}{2}$, and an even more complicated rule [equation (34)] for the case of spin 2. These equations involve implicitly the constant k, the determination of which is left to a later paper. The same method could theoretically be used for particles of even higher spin, though the commutation rules would soon become quite involved, and the details of deriving them laborious. *O. Frink* (State College, Pa.).

Knie, Guillermo. Wave mechanics in a curved space. Union Mat. Argentina. Memorias y Monografias (2) 1, 1–151 (1942). (Spanish. French summary) [MF 6769]

This monograph gives an expository and detailed account of some of the recent work of Schrödinger which attempts a fusion of the relativity and quantum theories. The principal topic dealt with is that of the transmission of electromagnetic and material waves in a gravitational field. The monograph appears to be based on the series of papers by Schrödinger appearing in the Proceedings of the Royal Irish Academy, v. 46, 1940–1941. *M. A. Basoco.*

Gross, B. Contribution to the theory of dielectric media. Anais Acad. Brasil. Sci. 12, 63–67 (1940). (Portuguese) [MF 6571]

In anomalous dielectrics the discharge potential $U(t)$ depends on the duration t_0 of the preceding charge. If the dielectric charged under a constant tension U_0, then, according to the theory of the author [cf. Phys. Rev. (2) 57, 57–59 (1940)], $R^{-1}U + CU_t(t) + \int_0^t \varphi(t-\tau)U_\tau(\tau)d\tau = -U_0\varphi(t+t_0)$, where $U_t = dU/dt$, R, C are constants and φ is a function characteristic of the dielectric, with $\varphi(\infty)=0$. The author gives a method of successive approximations for the solution of this equation and proves that $U_{t_0=0} = -RC(dU/dt)_{t_0=\infty}$. *I. Opatowski* (Minneapolis, Minn.).

Slepian, J. Energy and energy flow in the electromagnetic field. J. Appl. Phys. 13, 512–518 (1942).

A critical review of the usual way of deriving the Poynting vector and a new method of derivation are given. This yields other equally valid energy flow vectors, partly with the same, partly with different, divergences, and associated with other equally valid postulated electromagnetic energy densities. Several examples of such alternatives are given based not on purely formal, but definite physical concepts. Also, a definition is given which distinguishes between conduction and displacement currents in matter. It is shown that upon addition of a "magnetic" term, the energy flow as commonly defined in power engineering (that is, under quasistationary conditions), which is based on wattmeter readings, becomes a valid alternative Poynting vector. *H. G. Baerwald* (Cleveland, Ohio.)

Podolsky, Boris. A generalized electrodynamics. I. Non-quantum. Phys. Rev. (2) 62, 68–71 (1942). [MF 7021]

The author discusses a generalization of the Maxwell theory obtained by adding to the Lagrangian of that theory the invariant term

$$a^2\left(\sum_\beta \frac{\partial F_{\alpha\beta}}{\partial x^\beta}\right)^2,$$

where $F_{\alpha\beta}$ is the antisymmetric tensor formed from the field strengths. The resulting field equations involve the square of the four dimensional Laplacian operator. The solutions of these equations are discussed and compared with the results of Landé and Thomas [Phys. Rev. (2) 60, 514–523 (1941)]. *A. H. Taub* (Princeton, N. J.).

Schrödinger, Erwin. Non-linear optics. Proc. Roy. Irish Acad. Sect. A. 47, 77–117 (1942). [MF 6988]

This paper is, as a first installment, devoted to nonlinear electrodynamics in pursuance of the theory of M. Born [Proc. Roy. Soc. London. Ser. A. 143, 410–437 (1934)] and M. Born and L. Infeld [ibid. 144, 425–451 (1934)], which aims at removal of the "infinities" connected with the classical concept of point-singularities. A new outline (in complex version) of the basic Born-Infeld theory is given first, in which the differential equations themselves are linear, nonlinearity being introduced by a pair of supplementary conditions which are attached to the fundamental Lagrangian.

The first part deals with the mutual interaction of plane light waves involving weak fields (in the Born sense); it is a perturbation treatment, in general confined to the first approximation superposed on the zero order solution (Maxwell's theory) for very weak fields. In keeping with the complex formulation, circularly polarized waves are considered as fundamental components. It is found that interaction of two waves results in a decrease of propagation velocity proportional to the square of the amplitude of the interacting wave (in Born units) and to the fourth power of the sine of half the angle between the normals, but that two waves do not mutually scatter. The latter holds independently of the states of polarization and also in second approximation, and probably rigidly. (In quantum-mechanical treatment, scattering is due to multiple interaction with the nonobservable waves present with zero-point energy.) Three and more waves do scatter, and the associated quantitative expression for the products of scattering are derived, also those for energy density and Poynting vector of the two-wave system. An approximate treatment of blackbody

radiation, considered to be preliminary, gives a slightly enhanced energy flow of the hohlraum oscillator as compared with Maxwell's theory, the Planck law being corrected by the factor $3\cdot 10^{-48}T^4$.

The second part is devoted to the scattering of light waves by point charges and thus leaves the confine of "weak" fields. After consideration of the kind of anisotropy of free space induced by "strong" fields, the electric and magnetic polarizabilities are computed. Application is made to the scattering of long waves by point charges, purely electric and Rayleigh scattering being compounded; an outlook on the general problem of arbitrary wave-length is given. The results on polarizability permit a first-order correction of Coulomb's law. Finally the polarizability of the photon and an estimate of its scattering cross-section are derived. *H. G. Baerwald* (Cleveland, Ohio).

Silberstein, Ludwik. The effect of gradual light absorption in photographic exposure. J. Opt. Soc. Amer. **32**, 326–331 (1942). [MF 6698]

If φ is the photographic density (that is, $\varphi=\log(L_i-L_r)-\log L_t$, where L_i, L_r, L_t are, respectively, incident, reflected and transmitted light flux), ξ the exposure (that is, incident minus reflected light energy per unit area), a the thickness of the emulsion and k an absorption coefficient, then (*) $\varphi(\xi)=a^{-1}\int_0^a f(\xi e^{-kx})dx$. Physical considerations suggest the problem of finding f, under the condition $f(0)=0$, when φ is known. The differentiation of (*) gives (†) $f(\xi)-f(c\xi)=g(\xi)$, where $c=e^{-ka}$, $g(\xi)=ka\xi d\varphi/d\xi$. (†) is satisfied by the series $f(\xi)=\sum_{n=0}^{\infty}g(c^n\xi)$, convergent within the range of values which occur in applications. Any other solution of (†) is $f(\xi)+h(\xi)$, where $h(\xi)$ is any function satisfying $h(c\xi)=h(\xi)$. In the problem of the author $h\equiv 0$, from physical considerations. *I. Opatowski* (Chicago, Ill.).

Glaser, Walter. Strenge Berechnung magnetischer Linsen der Feldform $H=H_0/(1+(z/a)^2)$. Z. Phys. **117**, 285–315 (1941). [MF 6863]

Dosse, J. Strenge Berechnung magnetischer Linsen mit unsymmetrischer Feldform nach $H=H_0(1+(z/a)^2)$. Z. Phys. **117**, 316–321 (1941). [MF 6864]

Magnus, Wilhelm. Über die Beugung elektromagnetischer Wellen an einer Halbebene. Z. Phys. **117**, 168–179 (1941). [MF 6843]

H. Poincaré's plan of trying to calculate the current on a surface which diffracts electromagnetic waves is adopted for the two dimensional case. An integral equation

$$g(y)=\int_0^\infty f(\eta)H_0^{(2)}(|y-\eta|)d\eta$$

is obtained for the function $f(y)$ which is such that $f(\kappa\eta)$ gives the density and phase of the alternating surface current. The kernel has a logarithmic singularity and it seems natural to assume that $|f(\eta)|$ is infinite like $\eta^{-\frac{1}{2}}$ for small values of η, this behavior being like that of the electrostatic charge density at the edge of a charged half plane. To solve the integral equation expansions

$$g(y)=\sum_{n=0}^{\infty} i^n a_n J_n(y),$$

$$f(y)=\tfrac{1}{2}\pi e^{\frac{1}{4}i\pi}\sum_{m=0}^{\infty}(m+\tfrac{1}{2})i^{m-1}c_m y^{-1}J_{m+\frac{1}{2}}(y)$$

are adopted and these lead to a set of linear equations for the infinite set of coefficients c_m. A sufficient but not necessary condition for the existence of a solution is found to be that the series $\sum|a_n|$ converges. The case in which $g(y)=e^{-iy\sin\alpha}$ is then considered. A closed expression is finally found for the current density $f(\eta)$ and the transition to Sommerfeld's solution is discussed. *H. Bateman.*

Cherry, E. C. and Rivlin, R. S. Non-linear distortion, with particular reference to the theory of frequency modulated waves. II. Philos. Mag. (7) **33**, 272–293 (1942). [MF 6591]

In this second part of the paper, the methods developed in the first part [Philos. Mag. (7) **32**, 265–281 (1941); cf. these Rev. **3**, 160] are applied to devices having transmission characteristics which are nonlinear functions of frequency, with a carrier wave, modulated in any desired way, applied to the input terminals. The results obtained here are compared with those obtained by Carson and Fry [Bell System Tech. J. **16**, 513–540 (1937)], and certain discrepancies noted. *R. M. Foster* (New York, N. Y.).

Tellegen, B. D. H. Geometrical configurations and duality of electrical networks. Philips Tech. Rev. **5**, 324–330 (1940). [MF 7129]

Two electrical networks are said to be inverse if the volt-ampere relations of the one are isomorphic to the ampere-volt relations of the other. For a network without mutual inductance, an inverse exists (replacing resistance by conductance, capacitance by inductance and inductance by capacitance) if and only if the original network, considered as a topological graph, has a topological dual. It is an interesting matter to determine the conditions under which such dual networks exist, and the author settles this question by quoting two well-known theorems on topological graphs. The first, due to Hassler Whitney [Trans. Amer. Math. Soc. **34**, 339–362 (1932)], states that a graph has a dual if and only if it is planar, that is, can be imbedded in the plane without crossings. The second, due to Kuratowski [Fund. Math. **15**, 271–283 (1930)], asserts that a graph is planar unless it contains as a subgraph either (a) the "gas-water-electricity" configuration of two triples of points, each point of one triple joined to each point of the other triple, or (b) the configuration of five points, joined in all possible pairs. The problem of networks containing mutual inductances, which is much harder, is not treated.

H. Wallman (Cambridge, Mass.).

Grünberg, G. A., Kontorovitsch, M. I. und Lebedev, N. N. Über die zeitliche Entwicklung des Wärmedurchschlags fester Isolatoren. Acad. Sci. USSR. J. Phys. **5**, 339–356 (1941). [MF 6989]

The time variation of the heat transfer of a condenser is investigated theoretically. The mathematical formulation of the problem leads to a system of partial differential equations which are in part nonlinear. This system of differential equations is solved (a) by numerical integration and (b) by introducing certain approximations into these equations. The solutions (a) and (b) are shown to be in good agreement. *A. E. Heins* (Cambridge, Mass.).

Bewley, L. V. Traveling waves on electric power systems. Bull. Amer. Math. Soc. **48**, 527–538 (1942). [MF 7046]

An expository article describing the methods of attack used by the engineer on the mathematical problem of arbitrary surges on multiconductor transmission lines.

M. C. Gray (New York, N. Y.).

eces-
o be
hich
n is
ition
n.

with
ated
942).

oped
; cf.
ans-
s of
ired
ined
Fry
tain
).

ality
-330

volt-
pere-
tual
con-
by
ered
an
hich
ues-
gical
mer.
as a
d in
wski
ph is
gas-
ints,
ther
n all
tual

).

. N.
lags
-356

er is
tion
ntial
dif-
tion
hese
good
).

ems.
046]
tack
n of

).